

## ON INVERSION OF H-TRANSFORM IN $\mathcal{L}_{\nu,r}$ -SPACE

SERGEI A. SHLAPAKOV

Department of Mathematics  
Vitebsk Pedagogical University, Vitebsk 210036, Belarus

MEGUMI SAIGO

Department of Applied Mathematics  
Fukuoka University, Fukuoka 814-0180, Japan

ANATOLY A. KILBAS

Department of Mathematics and Mechanics  
Belarusian State University, Minsk 220050, Belarus

(Received December 1995 and in revised form September 25, 1997)

**ABSTRACT.** The paper is devoted to study the inversion of the integral transform

$$(\mathbf{H}f)(x) = \int_0^\infty H_{p,q}^{m,n} \left[ xt \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] f(t) dt$$

involving the  $H$ -function as the kernel in the space  $\mathcal{L}_{\nu,r}$  of functions  $f$  such that

$$\int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} < \infty \quad (1 < r < \infty, \nu \in \mathbb{R}).$$

**KEY WORDS AND PHRASES:**  $H$ -function, Integral transform,

**1991 AMS SUBJECT CLASSIFICATION CODES:** 33C40, 44A20

### 1. INTRODUCTION

This paper deals with the integral transforms of the form

$$(\mathbf{H}f)(x) = \int_0^\infty H_{p,q}^{m,n} \left[ xt \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] f(t) dt, \quad (1.1)$$

where  $H_{p,q}^{m,n} \left[ z \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right]$  is the  $H$ -function, which is a function of general hypergeometric type being introduced by S. Pincherle in 1888 (see [2, §1.19]). For integers  $m, n, p, q$  such that  $0 \leq m \leq q$ ,  $0 \leq n \leq p$ ,  $a_i, b_j \in \mathbb{C}$  and  $\alpha_i, \beta_j \in \mathbb{R}_+ = [0, \infty)$  ( $1 \leq i \leq p, 1 \leq j \leq q$ ), it can be

written by

$$\begin{aligned}
 H_{p,q}^{m,n} \left[ z \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] &= H_{p,q}^{m,n} \left[ z \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right. \right] \\
 &= \frac{1}{2\pi i} \int_L \mathcal{H}_{p,q}^{m,n} \left[ \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s \right] z^{-s} ds, \tag{1.2}
 \end{aligned}$$

where

$$\mathcal{H}_{p,q}^{m,n} \left[ \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s \right] = \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)}, \tag{1.3}$$

the contour  $L$  is specially chosen and an empty product, if it occurs, is taken to be one. The theory of this function may be found in Braaksma [1], Srivastava *et al.* [13, Chapter 1], Mathai and Saxena [8, Chapter 2] and Prudnikov *et al.* [9, §8.3]. We abbreviate the  $H$ -function (1.2) and the function (1.3) to  $H(z)$  and  $\mathcal{H}(s)$  when no confusion occurs. We note that the formal Mellin transform  $\mathfrak{M}$  of (1.1) gives the relation

$$(\mathfrak{M}Hf)(s) = \mathcal{H}(s)(\mathfrak{M}f)(1 - s). \tag{1.4}$$

Most of the known integral transforms can be put into the form (1.1), in particular, if  $\alpha_1 = \dots = \alpha_p = \beta_1 = \dots = \beta_q = 1$ , (1.1) is the integral transform with Meijer's  $G$ -function in the kernel (Rooney [11], Samko *et al.* [12, §36]). The integral transform (1.1) with the  $H$ -function kernel or the  $H$ -transform was investigated by many authors (see Bibliography in Kilbas *et al.* [5-6]). In Kilbas *et al.* [5-7] we have studied it in the space  $\mathcal{L}_{\nu,r}$  ( $1 \leq r < \infty$ ,  $\nu \in \mathbb{R}$ ) consisted of Lebesgue measurable complex valued functions  $f$  for which

$$\int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} < \infty. \tag{1.5}$$

We have investigated the mapping properties such as the boundedness, the representation and the range of the  $H$ -transform (1.1) on the space  $\mathcal{L}_{\nu,2}$  in Kilbas *et al.* [5] and on the space  $\mathcal{L}_{\nu,r}$  with any  $1 \leq r < \infty$  in Kilbas *et al.* [6-7], provided that  $a^* \geq 0$ ,  $\delta = 1$  and  $\Delta = 0$  or  $\Delta \neq 0$ , respectively. In Glaeske *et al.* [3] the results were extended to any  $\delta > 0$ . Here

$$a^* = \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j; \tag{1.6}$$

$$\delta = \prod_{i=1}^p \alpha_i^{-\alpha_i} \prod_{j=1}^q \beta_j^{\beta_j}; \tag{1.7}$$

$$\Delta = \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i. \tag{1.8}$$

In particular, we have proved that for certain ranges of parameters, the  $H$ -transform (1.1) have the representations

$$(Hf)(x) = hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H_{p+1,q+1}^{m,n+1} \left[ xt \left| \begin{matrix} (-\lambda, h), (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}, (-\lambda - 1, h) \end{matrix} \right. \right] f(t) dt \tag{1.9}$$

or

$$(Hf)(x) = -hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H_{p+1,q+1}^{m+1,n} \left[ xt \left| \begin{matrix} (a_i, \alpha_i)_{1,p}, (-\lambda, h) \\ (-\lambda - 1, h), (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] f(t) dt, \tag{1.10}$$

owing to the value of  $\text{Re}(\lambda)$ , where  $\lambda \in \mathbb{C}$  and  $h \in \mathbb{R} \setminus \{0\}$ .

In this paper we apply the results of Kilbas *et al.* [5-7] and Glaeske *et al.* [3] to find the inverse of the integral transforms (1.1) on the space  $\mathcal{L}_{\nu,r}$  with  $1 < r < \infty$  and  $\nu \in \mathbb{R}$ . Section 2 contains preliminary information concerning the properties of the  $H$ -transform (1.1) in the space  $\mathcal{L}_{\nu,r}$  and an asymptotic behavior of the  $H$ -function (1.2) at zero and infinity. In Sections 3 and 4 we prove that the inversion of the  $H$ -transform have the respective form (1.9) or (1.10):

$$f(x) = hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H_{p+1,q+1}^{q-m,p-n+1} \left[ xt \left| \begin{matrix} (-\lambda, h), (1-a_i-\alpha_i, \alpha_i)_{n+1,p}, (1-a_i-\alpha_i, \alpha_i)_{1,n} \\ (1-b_j-\beta_j, \beta_j)_{m+1,q}, (1-b_j-\beta_j, \beta_j)_{1,m}, (-\lambda-1, h) \end{matrix} \right. \right] (Hf)(t) dt \quad (1.11)$$

or

$$f(x) = -hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H_{p+1,q+1}^{q-m+1,p-n} \left[ xt \left| \begin{matrix} (1-a_i-\alpha_i, \alpha_i)_{n+1,p}, (1-a_i-\alpha_i, \alpha_i)_{1,n}, (-\lambda, h) \\ (-\lambda-1, h), (1-b_j-\beta_j, \beta_j)_{m+1,q}, (1-b_j-\beta_j, \beta_j)_{1,m} \end{matrix} \right. \right] (Hf)(t) dt, \quad (1.12)$$

provided that  $a^* = 0$ . Section 3 is devoted to treat on the spaces  $\mathcal{L}_{\nu,2}$  and  $\mathcal{L}_{\nu,r}$  with  $\Delta = 0$ , while Section 4 on the space  $\mathcal{L}_{\nu,r}$  with  $\Delta \neq 0$ .

The obtained results are extensions of those by Rooney [11] from  $G$ -transforms to  $H$ -transforms.

## 2. PRELIMINARIES

We give here some results from Kilbas *et al.* [5-6], Glaeske *et al.* [3] and from Kilbas and Saigo [4], Mathai and Saxena [8], Srivastava *et al.* [13] concerning the properties of  $H$ -transforms (1.1) in  $\mathcal{L}_{\nu,r}$ -spaces and the asymptotic behavior of the  $H$ -function at zero and infinity, respectively.

For the  $H$ -function (1.2), let  $a^*$  and  $\Delta$  be defined by (1.6) and (1.8) and let

$$\alpha = \begin{cases} \max \left[ -\text{Re} \left( \frac{b_1}{\beta_1} \right), \dots, -\text{Re} \left( \frac{b_m}{\beta_m} \right) \right] & \text{if } m > 0, \\ -\infty & \text{if } m = 0; \end{cases} \quad (2.1)$$

$$\beta = \begin{cases} \min \left[ \text{Re} \left( \frac{1-a_1}{\alpha_1} \right), \dots, \text{Re} \left( \frac{1-a_n}{\alpha_n} \right) \right] & \text{if } n > 0, \\ \infty & \text{if } n = 0; \end{cases} \quad (2.2)$$

$$a_1^* = \sum_{j=1}^m \beta_j - \sum_{i=n+1}^p \alpha_i; \quad a_2^* = \sum_{i=1}^n \alpha_i - \sum_{j=m+1}^q \beta_j; \quad a_1^* + a_2^* = a^*; \quad (2.3)$$

$$\mu = \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i + \frac{p-q}{2}. \quad (2.4)$$

For the function  $\mathcal{H}(s)$  given in (1.3), the exceptional set of  $\mathcal{H}$  is meant the set of real numbers  $\nu$  such that  $\alpha < 1-\nu < \beta$  and  $\mathcal{H}(s)$  has a zero on the line  $\text{Re}(s) = 1-\nu$  (see Rooney [11]). For two Banach space  $X$  and  $Y$  we denote by  $[X, Y]$  the collection of bounded linear operators from  $X$  to  $Y$ .

**THEOREM 2.1.** [5, Theorem 3], [6, Theorem 3.3] *Suppose that  $\alpha < 1-\nu < \beta$  and that either  $a^* > 0$  or  $a^* = 0$ ,  $\Delta(1-\nu) + \text{Re}(\mu) \leq 0$ . Then*

(a) *There is a one-to-one transform  $H \in [\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}]$  so that (1.4) holds for  $f \in \mathcal{L}_{\nu,2}$  and  $\text{Re}(s) = 1-\nu$ . If  $a^* = 0$ ,  $\Delta(1-\nu) + \text{Re}(\mu) = 0$  and  $\nu$  is not in the exceptional set of  $\mathcal{H}$ , then the operator  $H$  transforms  $\mathcal{L}_{\nu,2}$  onto  $\mathcal{L}_{1-\nu,2}$ .*

(b) If  $f \in \mathfrak{L}_{\nu,2}$  and  $\operatorname{Re}(\lambda) > (1 - \nu)h - 1$ ,  $\mathbf{H}f$  is given by (1.9). If  $f \in \mathfrak{L}_{\nu,2}$  and  $\operatorname{Re}(\lambda) < (1 - \nu)h - 1$ , then  $\mathbf{H}f$  is given by (1.10).

**THEOREM 2.2.** [6, Theorem 4.1], [3, Theorem 1] Let  $a^* = \Delta = 0, \operatorname{Re}(\mu) = 0$  and  $\alpha < 1 - \nu < \beta$ .

(a) The transform  $\mathbf{H}$  is defined on  $\mathfrak{L}_{\nu,2}$  and it can be extended to  $\mathfrak{L}_{\nu,r}$  as an element of  $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,r}]$  for  $1 < r < \infty$ .

(b) If  $1 < r \leq 2$ , the transform  $\mathbf{H}$  is one-to-one on  $\mathfrak{L}_{\nu,r}$  and there holds the equality

$$(\mathfrak{M}\mathbf{H}f)(s) = \mathcal{H}(s)(\mathfrak{M}f)(1 - s), \quad \operatorname{Re}(s) = 1 - \nu. \tag{2.5}$$

(c) If  $f \in \mathfrak{L}_{\nu,r}$  ( $1 < r < \infty$ ), then  $\mathbf{H}f$  is given by (1.9) for  $\operatorname{Re}(\lambda) > (1 - \nu)h - 1$ , while  $\mathbf{H}f$  is given by (1.10) for  $\operatorname{Re}(\lambda) < (1 - \nu)h - 1$ .

**THEOREM 2.3.** [6, Theorem 5.1], [3, Theorem 3] Let  $a^* = 0, \Delta > 0, -\infty < \alpha < 1 - \nu < \beta, 1 < r < \infty$  and  $\Delta(1 - \nu) + \operatorname{Re}(\mu) \leq 1/2 - \gamma(r)$ , where

$$\gamma(r) = \max \left[ \frac{1}{r}, \frac{1}{r'} \right] \quad \text{with} \quad \frac{1}{r} + \frac{1}{r'} = 1. \tag{2.6}$$

(a) The transform  $\mathbf{H}$  is defined on  $\mathfrak{L}_{\nu,2}$ , and it can be extended to  $\mathfrak{L}_{\nu,r}$  as an element of  $[\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,s}]$  for all  $s$  with  $r \leq s < \infty$  such that  $s' \geq [1/2 - \Delta(1 - \nu) - \operatorname{Re}(\mu)]^{-1}$  with  $1/s + 1/s' = 1$ .

(b) If  $1 < r \leq 2$ , the transform  $\mathbf{H}$  is one-to-one on  $\mathfrak{L}_{\nu,r}$  and there holds the equality (2.5).

(c) If  $f \in \mathfrak{L}_{\nu,r}$  and  $g \in \mathfrak{L}_{\nu,s}$  with  $1 < r < \infty, 1 < s < \infty, 1/r + 1/s \geq 1$  and  $\Delta(1 - \nu) + \operatorname{Re}(\mu) \leq 1/2 - \max[\gamma(r), \gamma(s)]$ , then the relation

$$\int_0^\infty f(x)(\mathbf{H}g)(x)dx = \int_0^\infty g(x)(\mathbf{H}f)(x)dx \tag{2.7}$$

holds.

The following two assertions give the asymptotic behavior of the the  $H$ -function (1.2) at zero and infinity provided that the poles of Gamma functions in the numerator of  $\mathcal{H}(s)$  do not coincide, i.e.

$$\beta_j(a_i - 1 - k) \neq \alpha_i(b_j + l) \quad (i = 1, \dots, n; j = 1, \dots, m; k, l = 0, 1, 2, \dots). \tag{2.8}$$

**THEOREM 2.4.** [8, §1.1.6], [13, §2.2] Let the condition (2.8) be satisfied and poles of Gamma functions  $\Gamma(b_j + \beta_j s)$  ( $j = 1, \dots, m$ ) be simple, i.e.

$$\beta_i(b_j + k) \neq \beta_j(b_i + l) \quad (i \neq j; i, j = 1, \dots, m; k, l = 0, 1, 2, \dots). \tag{2.9}$$

If  $\Delta \geq 0$ , then

$$H_{p,q}^{m,n}(z) = O(z^\rho) \quad (|z| \rightarrow 0) \quad \text{with} \quad \rho = \min_{1 \leq j \leq m} \left[ \frac{\operatorname{Re}(b_j)}{\beta_j} \right]. \tag{2.10}$$

**THEOREM 2.5.** [4, Corollary 3] Let  $a^*, \Delta$  and  $\mu$  be given by (1.6), (1.8) and (2.4), respectively. Let the conditions in (2.8) be satisfied and poles of Gamma functions  $\Gamma(1 - a_i - \alpha_i s)$  ( $i = 1, \dots, n$ ) be simple, i.e.

$$\alpha_j(1 - a_i + k) \neq \alpha_i(1 - a_j + l) \quad (i \neq j; i, j = 1, \dots, n; k, l = 0, 1, 2, \dots). \tag{2.11}$$

If  $a^* = 0$  and  $\Delta > 0$ , then

$$H_{p,q}^{m,n}(z) = O(z^\rho) \quad (|z| \rightarrow \infty) \quad \text{with} \quad \rho = \max \left[ \max_{1 \leq i \leq n} \left[ \frac{\text{Re}(a_i) - 1}{\alpha_i} \right], \frac{\text{Re}(\mu) + 1/2}{\Delta} \right]. \quad (2.12)$$

**REMARK 2.1.** It was proved in Kilbas and Saigo [4, §6] that if poles of Gamma functions  $\Gamma(1 - a_i - \alpha_i s)$  ( $i = 1, \dots, n$ ) are not simple (i.e. conditions in (2.11) are not satisfied), then the  $H$ -function (1.1) have power-logarithmic asymptotics at infinity. In this case the logarithmic multiplier  $[\log(z)]^N$  with  $N$  being the maximal number of orders of the poles may be added to the power multiplier  $z^\rho$  and hence the asymptotic estimate  $O(z^\rho)$  in (2.12) may be replaced by  $O(z^\rho [\log(z)]^N)$ . The same result is valid in the case of the asymptotics of the  $H$ -function (1.1) at zero, and the estimate  $O(z^\rho)$  in (2.10) may be replaced by  $O(z^\rho [\log(z)]^M)$ , where  $M$  is the maximal number of orders of the points at which the poles of  $\Gamma(b_j + \beta_j s)$  ( $j = 1, \dots, m$ ) coincide.

**3. INVERSION OF H-TRANSFORM IN  $\mathcal{L}_{\nu,2}$  AND  $\mathcal{L}_{\nu,r}$  WHEN  $\Delta = 0$**

In this and next sections we investigate that  $H$ -transform will have the inverse of the form (1.11) or (1.12). If  $f \in \mathcal{L}_{\nu,2}$ , and  $H$  is defined on  $\mathcal{L}_{\nu,r}$ , then according to Theorem 2.2, the equality (2.5) holds under the assumption there. This fact implies the relation

$$(\mathfrak{M}f)(s) = \frac{(\mathfrak{M}Hf)(1-s)}{\mathcal{H}(1-s)} \quad (3.1)$$

for  $\text{Re}(s) = \nu$ . By (1.3) we have

$$\frac{1}{\mathcal{H}(1-s)} = \mathcal{H}_{p,q}^{q-m,p-n} \left[ \begin{matrix} (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n} \\ (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m} \end{matrix} \middle| s \right] \equiv \mathcal{H}_0(s), \quad (3.2)$$

and hence (3.1) takes the form

$$(\mathfrak{M}f)(s) = (\mathfrak{M}Hf)(1-s)\mathcal{H}_0(s) \quad (\text{Re}(s) = \nu). \quad (3.3)$$

We denote by  $\alpha_0, \beta_0, a_0^*, a_{01}^*, a_{02}^*, \delta_0, \Delta_0$  and  $\mu_0$  for  $\mathcal{H}_0$  instead of those for  $\mathcal{H}$ . Then we find

$$\alpha_0 = \begin{cases} \max \left[ \frac{\text{Re}(b_{m+1}) - 1}{\beta_{m+1}} + 1, \dots, \frac{\text{Re}(b_q) - 1}{\beta_q} + 1 \right] & \text{if } q > m, \\ -\infty & \text{if } q = m; \end{cases} \quad (3.4)$$

$$\beta_0 = \begin{cases} \min \left[ \frac{\text{Re}(a_{n+1})}{\alpha_{n+1}} + 1, \dots, \frac{\text{Re}(a_p)}{\alpha_p} + 1 \right] & \text{if } p > n, \\ \infty & \text{if } p = n; \end{cases} \quad (3.5)$$

$$a_0^* = -a^*; \quad a_{01}^* = -a_2^*; \quad a_{02}^* = -a_1^*; \quad \delta_0 = \delta; \quad \Delta_0 = \Delta; \quad \mu_0 = -\mu - \Delta. \quad (3.6)$$

We also note that if  $\alpha_0 < \nu < \beta_0$ ,  $\nu$  is not in the exceptional set of  $\mathcal{H}_0$ .

First we consider the case  $r = 2$ .

**THEOREM 3.1.** Let  $\alpha < 1 - \nu < \beta, \alpha_0 < \nu < \beta_0, a^* = 0$  and  $\Delta(1 - \nu) + \text{Re}(\mu) = 0$ . If  $f \in \mathcal{L}_{\nu,2}$ , the relation (1.11) holds for  $\text{Re}(\lambda) > \nu h - 1$  and the relation (1.12) holds for  $\text{Re}(\lambda) < \nu h - 1$ .

**PROOF.** We apply Theorem 2.1 with  $\mathcal{H}$  being replaced by  $\mathcal{H}_0$  and  $\nu$  by  $1 - \nu$ . By the assumption and (3.6) we have

$$a_0^* = -a^* = 0, \tag{3.7}$$

$$\Delta_0[1 - (1 - \nu)] + \operatorname{Re}(\mu_0) = \Delta\nu - \operatorname{Re}(\mu) - \Delta = -[\Delta(1 - \nu) + \operatorname{Re}(\mu)] = 0 \tag{3.8}$$

and  $\alpha_0 < 1 - (1 - \nu) < \beta_0$ , and thus Theorem 2.1(a) applies. Then there is a one-to-one transform  $H_0 \in [\mathcal{L}_{1-\nu,2}, \mathcal{L}_{\nu,2}]$  so that the relation

$$(\mathfrak{M}H_0f)(s) = \mathcal{H}_0(s)(\mathfrak{M}f)(1 - s) \tag{3.9}$$

holds for  $f \in \mathcal{L}_{1-\nu,2}$  and  $\operatorname{Re}(s) = \nu$ . Further if  $f \in \mathcal{L}_{\nu,2}$ ,  $Hf \in \mathcal{L}_{1-\nu,2}$  and it follows from (3.9), (1.4) and (3.2) that

$$(\mathfrak{M}H_0Hf)(s) = \mathcal{H}_0(s)(\mathfrak{M}Hf)(1 - s) = \mathcal{H}_0(s)\mathcal{H}(1 - s)(\mathfrak{M}f)(s) = (\mathfrak{M}f)(s),$$

if  $\operatorname{Re}(s) = \nu$ . Hence  $\mathfrak{M}H_0Hf = \mathfrak{M}f$  and

$$H_0Hf = f \quad \text{for } f \in \mathcal{L}_{\nu,2}. \tag{3.10}$$

Applying Theorem 2.1(b) with  $\mathcal{H}$  being replaced by  $\mathcal{H}_0$  and  $\nu$  by  $1 - \nu$ , we obtain for  $f \in \mathcal{L}_{1-\nu,2}$  that

$$\begin{aligned} (H_0f)(x) &= hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \\ &\cdot \int_0^\infty H_{p+1,q+1}^{q-m,p-n+1} \left[ xt \left| \begin{matrix} (-\lambda, h), (1-a, -\alpha_i, \alpha_i)_{n+1,p}, (1-a, -\alpha_i, \alpha_i)_{1,n} \\ (1-b_j - \beta_j, \beta_j)_{m+1,q}, (1-b_j - \beta_j, \beta_j)_{1,m}, (-\lambda-1, h) \end{matrix} \right. \right] f(t) dt, \end{aligned} \tag{3.11}$$

if  $\operatorname{Re}(\lambda) > [1 - (1 - \nu)]h - 1$  and

$$\begin{aligned} (H_0f)(x) &= -hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \\ &\cdot \int_0^\infty H_{p+1,q+1}^{q-m+1,p-n} \left[ xt \left| \begin{matrix} (1-a_i - \alpha_i, \alpha_i)_{n+1,p}, (1-a_i - \alpha_i, \alpha_i)_{1,n}, (-\lambda, h) \\ (-\lambda-1, h), (1-b_j - \beta_j, \beta_j)_{m+1,q}, (1-b_j - \beta_j, \beta_j)_{1,m} \end{matrix} \right. \right] f(t) dt, \end{aligned} \tag{3.12}$$

if  $\operatorname{Re}(\lambda) < [1 - (1 - \nu)]h - 1$ . Replacing  $f$  by  $Hf$  and using (3.10) we have the relations (1.11) and (1.12) for  $f \in \mathcal{L}_{\nu,2}$ , if  $\operatorname{Re}(\lambda) > \nu h - 1$  and  $\operatorname{Re}(\lambda) < \nu h - 1$ , respectively, which completes the proof of theorem.

Next results is the extension of Theorem 3.1 to  $\mathcal{L}_{\nu,r}$ -spaces for any  $1 < r < \infty$ , provided that  $\Delta = 0$  and  $\operatorname{Re}(\mu) = 0$ .

**THEOREM 3.2.** *Let  $\alpha < 1 - \nu < \beta, \alpha_0 < \nu < \beta_0, a^* = 0, \Delta = 0$  and  $\operatorname{Re}(\mu) = 0$ . If  $f \in \mathcal{L}_{\nu,r}$  ( $1 < r < \infty$ ), the relation (1.11) holds for  $\operatorname{Re}(\lambda) > \nu h - 1$  and the relation (1.12) holds for  $\operatorname{Re}(\lambda) < \nu h - 1$ .*

**PROOF.** We apply Theorem 2.2 with  $\mathcal{H}$  being replaced by  $\mathcal{H}_0$  and  $\nu$  by  $\nu - 1$ . By the assumption and (3.6), we have  $a_0^* = \Delta_0 = 0, \operatorname{Re}(\mu_0) = 0$  and  $\alpha_0 < 1 - (1 - \nu) < \beta_0$ , and thus Theorem 2.2(a) can be applied. In accordance with this theorem,  $H_0$  can be extended to  $\mathcal{L}_{1-\nu,r}$  as an element of  $H_0 \in [\mathcal{L}_{1-\nu,r}, \mathcal{L}_{\nu,r}]$ . By virtue of (3.10)  $H_0H$  is identical operator in  $\mathcal{L}_{\nu,2}$ . By Rooney [11, Lemma 2.2]  $\mathcal{L}_{\nu,2}$  is dense in  $\mathcal{L}_{\nu,r}$  and since  $H \in [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}]$  and  $H_0 \in [\mathcal{L}_{1-\nu,r}, \mathcal{L}_{\nu,r}]$ , the operator  $H_0H$  is identical in  $\mathcal{L}_{\nu,r}$  and hence

$$H_0Hf = f \quad \text{for } f \in \mathcal{L}_{\nu,r}. \tag{3.13}$$

Applying Theorem 2.2(c) with  $\mathcal{H}$  being replaced by  $\mathcal{H}_0$  and  $\nu$  by  $1 - \nu$ , we obtain that the relations (3.11) and (3.12) hold for  $f \in \mathcal{L}_{1-\nu,r}$ , when  $\text{Re}(\lambda) > [1 - (1 - \nu)]h - 1$  and  $\text{Re}(\lambda) < [1 - (1 - \nu)]h - 1$ , respectively. Replacing  $f$  by  $Hf$  and using (3.13), we arrive at (1.11) and (1.12) for  $f \in \mathcal{L}_{1-\nu,r}$ , if  $\text{Re}(\lambda) > \nu h - 1$  and  $\text{Re}(\lambda) < \nu h - 1$ , respectively, which completes the proof of theorem.

**REMARK 3.1.** If  $\alpha_1 = \dots = \alpha_p = \beta_1 = \dots = \beta_q = 1$  which means that the  $H$ -function (1.2) is Meijer's  $G$ -function, then  $\Delta = q - p$  and Theorems 8.1 and 8.2 in Rooney [11] follow from Theorems 3.1 and 3.2.

**4. INVERSION OF H-TRANSFORM IN  $\mathcal{L}_{\nu,r}$  WHEN  $\Delta \neq 0$**

We now investigate under what condition the  $H$ -transform with  $\Delta \neq 0$  will have the inverse of the form (1.11) or (1.12). First, we consider the case  $\Delta > 0$ . To obtain the inversion of the  $H$ -transform on  $\mathcal{L}_{\nu,r}$  we use the relation (2.7).

**THEOREM 4.1.** Let  $1 < r < \infty, -\infty < \alpha < 1 - \nu < \beta, \alpha_0 < \nu < \min\{\beta_0, [\text{Re}(\mu + 1/2)/\Delta] + 1\}, a^* = 0, \Delta > 0$  and  $\Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r)$ , where  $\gamma(r)$  is given in (2.6). If  $f \in \mathcal{L}_{\nu,r}$ , then the relations (1.11) and (1.12) hold for  $\text{Re}(\lambda) > \nu h - 1$  and for  $\text{Re}(\lambda) < \nu h - 1$ , respectively.

**PROOF.** According to Theorem 2.3(a), the  $H$ -transform is defined on  $\mathcal{L}_{\nu,r}$ . First we consider the case  $\text{Re}(\lambda) > \nu h - 1$ . Let  $H_1(t)$  be the function

$$H_1(t) = H_{p+1,q+1}^{q-m,p-n+1} \left[ t \left| \begin{matrix} (-\lambda, h), (1 - a_i - \alpha_i, \alpha_i)_{n+1,p}, (1 - a_i - \alpha_i, \alpha_i)_{1,n} \\ (1 - b_j - \beta_j, \beta_j)_{m+1,q}, (1 - b_j - \beta_j, \beta_j)_{1,m}, (-\lambda - 1, h) \end{matrix} \right. \right]. \tag{4.1}$$

If we denote by  $\tilde{a}^*, \tilde{\delta}, \tilde{\Delta}$  and  $\tilde{\mu}$  for  $H_1$  instead of those for  $H$ , then

$$\tilde{a}^* = -a^* = 0; \quad \tilde{\delta} = \delta; \quad \tilde{\Delta} = \Delta > 0; \quad \tilde{\mu} = -\mu - \Delta - 1. \tag{4.2}$$

We prove that  $H_1 \in \mathcal{L}_{\nu,s}$  for any  $s$  ( $1 \leq s < \infty$ ). For this, we first apply Theorems 2.4 and 2.5 and Remark 2.1 to  $H_1(t)$  to find its asymptotic behavior at zero and infinity. According to (3.4), (3.5) and the assumptions, we find

$$\begin{aligned} \frac{\text{Re}(b_j) - 1}{\beta_j} + 1 &\leq \alpha_0 < \beta_0 \leq \frac{\text{Re}(a_i)}{\alpha_i} + 1 \quad (j = m + 1, \dots, q; i = n + 1, \dots, p); \\ \frac{\text{Re}(b_j) - 1}{\beta_j} + 1 &\leq \alpha_0 < \nu < \frac{\text{Re}(\lambda) + 1}{h} \quad (j = m + 1, \dots, q). \end{aligned}$$

Then it follows from here that the poles

$$a_{ik} = \frac{a_i + k}{\alpha_i} + 1 \quad (i = n + 1, \dots, p; k = 0, 1, 2, \dots), \quad \lambda_n = \frac{\lambda + 1 + n}{h} \quad (n = 0, 1, 2, \dots)$$

of Gamma functions  $\Gamma(a_i + \alpha_i - \alpha_i s)$  ( $i = n + 1, \dots, p$ ) and  $\Gamma(1 + \lambda - hs)$ , and the poles

$$b_{jl} = \frac{b_j - 1 - l}{\beta_j} + 1 \quad (j = m + 1, \dots, q; l = 0, 1, 2, \dots)$$

of Gamma functions  $\Gamma(1 - b_j - \beta_j + \beta_j s)$  ( $j = m + 1, \dots, q$ ) do not coincide. Hence by Theorem 2.4, (4.1) and Remark 2.1, we have

$$H_1(t) = O(t^{\rho_1}) \quad (|t| \rightarrow 0) \quad \text{with} \quad \rho_1 = \min_{m+1 \leq j \leq q} \left[ \frac{1 - \text{Re}(b_j)}{\beta_j} \right] - 1 = -\alpha_0$$

for  $\alpha_0$  being given in (3.4), or

$$H_1(t) = O(t^{-\alpha_0}) \quad (t \rightarrow 0) \tag{4.3}$$

with an additional logarithmic multiplier  $[\log t]^N$  possibly, if Gamma functions  $\Gamma(1 - b_j - \beta_j + \beta_j s)$  ( $j = m + 1, \dots, q$ ) have general poles of order  $N \geq 2$  at some points.

Further by Theorem 2.5, (4.1) and Remark 2.1,

$$H_1(t) = O(t^{\varrho_1}) \quad (t \rightarrow \infty) \quad \text{with} \quad \varrho_1 = \max \left[ \beta_0, \frac{-\text{Re}(\mu) - 1/2}{\Delta} - 1, \frac{-\text{Re}(\lambda) - 1}{h} \right]$$

for  $\beta_0$  being given by (3.5), or

$$H_1(t) = O(t^{-\gamma_0}) \quad (|t| \rightarrow \infty) \quad \text{with} \quad \gamma_0 = \min \left[ \beta_0, \frac{\text{Re}(\mu) + 1/2}{\Delta} + 1, \frac{\text{Re}(\lambda) + 1}{h} \right] \tag{4.4}$$

and with an additional logarithmic multiplier  $[\log(t)]^M$  possibly, if Gamma functions  $\Gamma(1 + \lambda - hs), \Gamma(a_i + \alpha_i - \alpha_i s)$  ( $i = n + 1, \dots, p$ ) have general poles of order  $M \geq 2$  at some points.

Let Gamma functions  $\Gamma(1 - b_j - \beta_j + \beta_j s)$  ( $j = m + 1, \dots, q$ ) and  $\Gamma(1 + \lambda - hs), \Gamma(a_i + \alpha_i - \alpha_i s)$  ( $i = n + 1, \dots, p$ ) have simple poles. Then from (4.3) and (4.4) we see that for  $1 \leq s < \infty$ ,  $H_1(t) \in \mathcal{L}_{\nu, s}$  if and only if, for some  $R_1$  and  $R_2$ ,  $0 < R_1 < R_2 < \infty$ , the integrals

$$\int_0^{R_1} t^{s(\nu - \alpha_0) - 1} dt, \quad \int_{R_2}^{\infty} t^{s(\nu - \gamma_0) - 1} dt \tag{4.5}$$

are convergent. Since by the assumption  $\nu > \alpha_0$ , the first integral in (4.5) converges. In view of our assumptions

$$\nu < \beta_0, \quad \nu < \frac{\text{Re}(\mu) + 1/2}{\Delta} + 1, \quad \nu < \frac{\text{Re}(\lambda) + 1}{h}$$

we find  $\nu - \gamma_0 < 0$  and the second integral in (4.5) converges, too.

If Gamma functions  $\Gamma(1 - b_j - \beta_j + \beta_j s)$  ( $j = m + 1, \dots, q$ ) or  $\Gamma(1 + \lambda - hs), \Gamma(a_i + \alpha_i - \alpha_i s)$  ( $i = n + 1, \dots, p$ ) have general poles, then the logarithmic multipliers  $[\log(t)]^N$  ( $N = 1, 2, \dots$ ) may be added in the integrals in (4.5), but they do not influence on the convergence of them. Hence, under the assumptions we have

$$H_1(t) \in \mathcal{L}_{\nu, s} \quad (1 \leq s < \infty). \tag{4.6}$$

Let  $a$  be a positive number and  $\Pi_a$  denote the operator

$$(\Pi_a f)(x) = f(ax) \quad (x > 0) \tag{4.7}$$

for a function  $f$  defined almost everywhere on  $(0, \infty)$ . It is known in Rooney [11, p.268] that  $\Pi_a$  is a bounded isomorphism of  $\mathcal{L}_{\nu, r}$  onto  $\mathcal{L}_{a\nu, r}$ , and if  $f \in \mathcal{L}_{\nu, r}$  ( $1 \leq r \leq 2$ ), there holds the relation for the Mellin transform  $\mathfrak{M}$

$$(\mathfrak{M}\Pi_a f)(s) = a^{-s}(\mathfrak{M}f)\left(\frac{s}{a}\right) \quad (\text{Re}(s) = \nu). \tag{4.8}$$

By virtue of Theorem 2.3(c) and (4.6), if  $f \in \mathcal{L}_{\nu, r}$  and  $H_1 \in \mathcal{L}_{\nu, r'}$  (and hence  $\Pi_x H_1 \in \mathcal{L}_{\nu, r'}$ ), then

$$\int_0^{\infty} H_1(xt)(\mathbf{H}f)(t)dt = \int_0^{\infty} (\Pi_x H_1)(t)(\mathbf{H}f)(t)dt = \int_0^{\infty} (\mathbf{H}\Pi_x H_1)(t)f(t)dt. \tag{4.9}$$

From the assumption  $\Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \leq 0$ , Theorem 2.3(b) and (4.8) imply that

$$(\mathfrak{M}\mathbf{H}\Pi_x H_1)(s) = \mathcal{H}(s)(\mathfrak{M}\Pi_x H_1)(1 - s) = x^{-(1-s)}\mathcal{H}(s)(\mathfrak{M}H_1)(1 - s) \tag{4.10}$$



for  $\text{Re}(s) = 1 - \nu$ . Now from (4.6),  $H_1(t) \in \mathcal{L}_{\nu,1}$ . Then by the definitions of the  $H$ -function (1.2), (1.3) and the direct and inverse Mellin transforms (see, for example, Samko *et al.* [12, (1.112), (1.113)]), we have

$$\begin{aligned} (\mathfrak{M}H_1)(s) &= \mathcal{H}_{p+1,q+1}^{q-m,p-n+1} \left[ \begin{matrix} (-\lambda, h), (1-a_i-\alpha_i, \alpha_i)_{n+1,p}, (1-a_i-\alpha_i, \alpha_i)_{1,n} \\ (1-b_j-\beta_j, \beta_j)_{m+1,q}, (1-b_j-\beta_j, \beta_j)_{1,m}, (-\lambda-1, h) \end{matrix} \middle| s \right] \\ &= \mathcal{H}_{p,q}^{q-m,p-n} \left[ \begin{matrix} (1-a_i-\alpha_i, \alpha_i)_{n+1,p}, (1-a_i-\alpha_i, \alpha_i)_{1,n} \\ (1-b_j-\beta_j, \beta_j)_{m+1,q}, (1-b_j-\beta_j, \beta_j)_{1,m} \end{matrix} \middle| s \right] \frac{\Gamma(1+\lambda-hs)}{\Gamma(2+\lambda-hs)} \\ &= \frac{\mathcal{H}_0(s)}{1+\lambda-hs} \end{aligned}$$

for  $\text{Re}(s) = \nu$ , where  $\mathcal{H}_0$  is given by (3.2). It follows from here that for  $\text{Re}(s) = 1 - \nu$ ,

$$(\mathfrak{M}H_1)(1-s) = \frac{\mathcal{H}_0(1-s)}{1+\lambda-h(1-s)} = \frac{1}{\mathcal{H}(s)[1+\lambda-h(1-s)]}.$$

Substituting this into (4.10) we obtain

$$(\mathfrak{M}H\Pi_x H_1)(s) = \frac{x^{-(1-s)}}{1+\lambda-h(1-s)} \quad (\text{Re}(s) = 1 - \nu). \tag{4.11}$$

For  $x > 0$  let us denote by  $g_x(t)$  a function

$$g_x(t) = \begin{cases} \frac{1}{h} t^{(\lambda+1)/h-1} & \text{if } 0 < t < x, \\ 0 & \text{if } t > x, \end{cases} \tag{4.12}$$

then

$$(\mathfrak{M}g_x)(s) = \frac{x^{s+(\lambda+1)/h-1}}{1+\lambda-h(1-s)},$$

and (4.11) takes the form

$$(\mathfrak{M}H\Pi_x H_1)(s) = (\mathfrak{M}[x^{-(\lambda+1)/h} g_x])(s),$$

which implies

$$(H\Pi_x H_1)(t) = x^{-(\lambda+1)/h} g_x(t). \tag{4.13}$$

Substituting (4.13) into (4.9), we have

$$\int_0^\infty H_1(xt)(Hf)(t)dt = x^{-(\lambda+1)/h} \int_0^\infty g_x(t)f(t)dt$$

or, in accordance with (4.12),

$$\int_0^x t^{(\lambda+1)/h-1} f(t)dt = hx^{(\lambda+1)/h} \int_0^\infty H_1(xt)(Hf)(t)dt.$$

Differentiating this relation we obtain

$$f(x) = hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H_1(xt)(Hf)(t)dt$$

which shows (1.11).

If  $\text{Re}(\lambda) < \nu h - 1$ , the relation (1.12) is proved similarly to (1.11), by taking the function

$$H_2(t) = \mathcal{H}_{p+1,q+1}^{q-m+1,p-n} \left[ t \left[ \begin{matrix} (1-a_i-\alpha_i, \alpha_i)_{n+1,p}, (1-a_i-\alpha_i, \alpha_i)_{1,n}, (-\lambda, h) \\ (-\lambda-1, h), (1-b_j-\beta_j, \beta_j)_{m+1,q}, (1-b_j-\beta_j, \beta_j)_{1,m} \end{matrix} \right] \right] \tag{4.14}$$

instead of the function  $H_1(t)$  in (4.1). This completes the proof of the theorem.

In the case  $\Delta < 0$  the following statement gives the inversion of  $H$ -transform on  $\mathcal{L}_{\nu,r}$ .

**THEOREM 4.2.** *Let  $1 < r < \infty$ ,  $\alpha < 1 - \nu < \beta < \infty$ ,  $\max\{\alpha_0, \{\operatorname{Re}(\mu + 1/2)/\Delta\} + 1\} < \nu < \beta_0$ ,  $\alpha^* = 0$ ,  $\Delta < 0$  and  $\Delta(1 - \nu) + \operatorname{Re}(\mu) \leq 1/2 - \gamma(r)$ , where  $\gamma(r)$  is given by (2.6). If  $f \in \mathcal{L}_{\nu,r}$ , then the relations (1.11) and (1.12) holds for  $\operatorname{Re}(\lambda) > \nu h - 1$  and for  $\operatorname{Re}(\lambda) < \nu h - 1$ , respectively.*

This theorem is proved similarly to Theorem 4.1, if we apply Theorem 5.2 from Kilbas *et al.* [6] instead of Theorem 2.3 and take into account the asymptotics of the  $H$ -function at zero and infinity (see Srivastava *et al.* [13, §2.2] and Kilbas and Saigo [4, Corollary 4]).

**REMARK 4.1.** If  $\alpha_1 = \dots = \alpha_p = \beta_1 = \dots = \beta_q = 1$ , then Theorems 8.3 and 8.4 in Rooney [11] follow from Theorems 4.1 and 4.2.

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