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SUBCONTRA-CONTINUOUS FUNCTIONS

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ABSTRACT. A weak form of contra-continuity, called subcontra-continuity, is introduced. It is shown that subcontra-continuity is strictly weaker than contra-continuity and stronger than both subweak continuity and sub-LC-continuity. Subcontra-continuity is used to improve several results in the literature concerning compact spaces.

KEY WORDS AND PHRASES: subcontra-continuity, contra-continuity, subweak continuity, sub-LC-continuity.
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1. INTRODUCTION

In [1] Dontchev introduced the notion of a contra-continuous function. In this note we develop a weak form of contra-continuity, which we call subcontra-continuity. We show that subcontra-continuity implies both subweak continuity and sub-LC-continuity. We also establish some of the properties of subcontra-continuous functions. In particular it is shown that the graph of a subcontra-continuous function into a T_1-space is closed. Finally, we show that many of the applications of contra-continuous functions to compact spaces established by Dontchev [1] hold for subcontra-continuous functions. For example, we establish that the subcontra-continuous, nearly continuous image of an almost compact space is compact and that the subcontra-continuous, β-continuous image of an S-closed space is compact.

2. PRELIMINARIES

The symbols X and Y denote topological spaces with no separation axioms assumed unless explicitly stated. The closure and interior of a subset A of a space X are signified by $Cl(A)$ and $Int(A)$, respectively. A set A is regular open (semi-open, nearly open) provided that $A = Int(Cl(A))$ ($A \subseteq Cl(Int(A))$, $A \subseteq Int(Cl(A))$) and A is regular closed (semi-closed) if its complement is regular open (semi-open). A set A is locally closed provided that $A = U \cap F$, where U is an open set and F is a closed set.

DEFINITION 1. Dontchev [1]. A function $f : X \rightarrow Y$ is said to contra-continuous provided that for every open set V in Y, $f^{-1}(V)$ is closed in X.

DEFINITION 2. Rose [2]. A function $f : X \rightarrow Y$ is said to be subweakly continuous if there is an open base $\mathcal{B}$ for the topology on Y such that $Cl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every $V \in \mathcal{B}$.

DEFINITION 3. Ganster and Reilly [3]. A function $f : X \to Y$ is said to be sub-LC-continuous provided there is an open base $\mathcal{B}$ for the topology on Y such that $f^{-1}(V)$ is locally closed for every $V \in \mathcal{B}$.

DEFINITION 4. A function $f : X \to Y$ is said to be semi-continuous (Levine [4]) (nearly continuous (Ptak [5]), β-continuous (Abd El-Monsef *et al.* [6])) if for every open set V in Y, $f^{-1}(V) \subseteq Cl(Int(f^{-1}(V)))$ ($f^{-1}(V) \subseteq Int(Cl(f^{-1}(V)))$, $f^{-1}(V) \subseteq Cl(Int(Cl(f^{-1}(V))))$).

DEFINITION 5. Gentry and Hoyle [7]. A function $f : X \to Y$ is said to be c-continuous if, for every $x \in X$ and every open set V in Y containing $f(x)$ and with compact complement, there exists an open set U in X containing x such that $f(U) \subseteq V$.

3. SUBCONTRA-CONTINUOUS FUNCTIONS

We define a function $f : X \to Y$ to be subcontra-continuous provided there exists an open base $\mathcal{B}$ for the topology on Y such that $f^{-1}(V)$ is closed in X for every $V \in \mathcal{B}$. Obviously contra-continuity implies subcontra-continuity. The following example shows that the reverse implication does not hold.

EXAMPLE 1. Let X be a nondiscrete T_1-space and let Y be the set X with the discrete topology. Finally let $f : X \to Y$ be the identity mapping. If $\mathcal{B}$ is the collection of all singleton subsets of Y, then $\mathcal{B}$ is an open base for the topology on Y. Since X is T_1, f is subcontra-continuous with respect to $\mathcal{B}$. Obviously f is not contra-continuous.

Subcontra-continuity is independent of continuity. The function in Example 1 is subcontra-continuous but not continuous. The next example shows that continuity does not imply subcontra-continuity.

EXAMPLE 2 Let $X = \{a, b\}$ be the Sierpinski space with the topology $\mathcal{T} = \{X, \emptyset, \{a\}\}$ and let $f : X \to X$ be the identity mapping. Obviously f is continuous. However, any open base for the topology on X must contain $\{a\}$ and $f^{-1}(\{a\})$ is not closed. It follows that f is not subcontra-continuous.

Since closed sets are locally closed, subcontra-continuity implies sub-LC-continuity. We see from the following theorem that subcontra-continuity also implies subweak continuity.

THEOREM 1. Every subcontra-continuous function is subweakly continuous.

PROOF. Assume $f : X \to Y$ is subcontra-continuous. Let $\mathcal{B}$ be an open base for the topology on Y for which $f^{-1}(V)$ is closed in X for every $V \in \mathcal{B}$. Then for $V \in \mathcal{B}$, $Cl(f^{-1}(V)) = f^{-1}(V) \subseteq f^{-1}(Cl(V))$ and hence f is subweakly continuous. □

Since a subweakly continuous function into a Hausdorff space has a closed graph (Baker [8]), a subcontra-continuous function into a Hausdorff space has a closed graph. However, the following stronger result holds for subcontra-continuous functions.

THEOREM 2. If $f : X \to Y$ is a subcontra-continuous function and Y is T_1, then the graph of f, $G(f)$, is closed.

PROOF. Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Let $\mathcal{B}$ be an open base for the topology on Y for which $f^{-1}(V)$ is closed in X for every $V \in \mathcal{B}$. Since Y is T_1, there exists $V \in \mathcal{B}$ such that $y \in V$ *and* $f(x) \notin V$. Then we see that $(x, y) \in (X - f^{-1}(V)) \times V \subseteq X \times Y - G(f)$. It follows that $G(f)$ is closed. □

COROLLARY 1. If $f : X \to Y$ is contra-continuous and Y is T_1, then the graph of f is closed.

Long and Hendrix [9] proved that the closed graph property implies c-continuity. Therefore we have the following corollary.

COROLLARY 2. If $f : X \to Y$ is subcontra-continuous and Y is T_1, then f is c-continuous.

The next two results are also implied by the closed graph property (Fuller [10]).

COROLLARY 3. If $f : X \to Y$ is subcontra-continuous and Y is T_1, then for every compact subset C of Y, $f^{-1}(C)$ is closed in X.

COROLLARY 4. If $f : X \to Y$ is subcontra-continuous and Y is T_1, then for every compact subset C of X, $f(C)$ is closed.

For a function $f : X \to Y$, the graph function of f is the function $g : X \to X \times Y$ given by $g(x) = (x, f(x))$. We shall see in the following example that the graph function of a subcontra-continuous function is not necessarily subcontra-continuous.

EXAMPLE 3. Let $X = \{a, b\}$ be the Sierpinski space with the topology $\mathcal{T} = \{X, \emptyset, \{a\}\}$ and let $f : X \to X$ be given by $f(a) = b$ and $f(b) = a$. Obviously f is subcontra-continuous, in fact contra-continuous. Let $\mathcal{B}$ be any open base for the product topology on $X \times Y$. Then there exists $V \in \mathcal{B}$ for which $(a, b) \in V \subseteq \{(a, a), (a, b)\}$. We see that $V = \{(a, a), (a, b)\}$ and that, if $g : X \to X \times X$ is the graph function for f, then $g^{-1}(V) = \{a\}$ which is not closed. Thus the graph function of f is not subcontra-continuous.

However, the following result does hold for the graph function.

THEOREM 3. The graph function of a subcontra-continuous function is sub-LC-continuous.

PROOF. Assume $f : X \to Y$ is subcontra-continuous and let $g : X \to X \times Y$ be the graph function of f. Let $\mathcal{B}$ be an open base for the topology on Y for which $f^{-1}(V)$ is closed in X for every $V \in \mathcal{B}$. Then $\{U \times V : U \text{ is open in } X, V \in \mathcal{B}\}$ is an open base for the product topology on $X \times Y$. Since $g^{-1}(U \times V) = U \cap f^{-1}(V)$, we see that g is sub-LC-continuous. □

The graph function of a subweakly continuous function is subweakly continuous (Baker [8]) and the graph function of a sub-LC-continuous function is sub-LC-continuous (Ganster and Reilly [3]). It follows that the graph function in Example 3 is subweakly continuous and sub-LC-continuous but not subcontra-continuous. Therefore subcontra-continuity is strictly stronger than sub-LC-continuity and subweak continuity.

THEOREM 4. If Y is a T_1-space and $f : X \to Y$ is a subcontra-continuous injection, then X is T_1.

PROOF. Let x_1 and x_2 be distinct points in X. Let $\mathcal{B}$ be an open base for the topology on Y for which $f^{-1}(V)$ is closed in X for every $V \in \mathcal{B}$. Since Y is T_1 and $f(x_1) \neq f(x_2)$, there exists $V \in \mathcal{B}$ such that $f(x_1) \notin V$ and $f(x_2) \in V$. Then $x_1 \in X - f^{-1}(V)$ which is open and $x_2 \notin X - f^{-1}(V)$. □

THEOREM 5. Let $A \subseteq X$ and $f : X \to X$ be a subcontra-continuous function such that $f(X) = A$ and $f|_A$ is the identity on A. Then, if X is T_1, A is closed in X.

PROOF. Suppose A is not closed. Let $x \in Cl(A) - A$. Let $\mathcal{B}$ be an open base for the topology on Y for which $f^{-1}(V)$ is closed for every $V \in \mathcal{B}$. Since $x \notin A$, we have that $x \neq f(x)$. Since X is T_1, there exists $V \in \mathcal{B}$ such that $x \in V$ and $f(x) \notin V$. Let U be an open set containing x. Then $x \in U \cap V$ which is open. Since $x \in Cl(A)$, $(U \cap V) \cap A \neq \emptyset$. Let $y \in (U \cap V) \cap A$. Since $y \in A$, $f(y) = y \in V$. So $y \in f^{-1}(V)$. Thus $y \in U \cap f^{-1}(V)$ and hence $U \cap f^{-1}(V) \neq \emptyset$. We see that $x \in Cl(f^{-1}(V)) = f^{-1}(V)$ which is a contradiction. Therefore A is closed. □

The next result follows easily for the definition.

THEOREM 6. If $f : X \to Y$ is subcontra-continuous, then for every open set V in Y, $f^{-1}(V)$ is a union of closed sets in X.

Obviously every function with a T_1-domain satisfies the above condition. However, as we see in the following example, a function with a T_1-domain can fail to be subcontra-continuous. It follows that the converse of Theorem 6 does not hold..

EXAMPLE 4. Let $X = \mathbb{R}$ with the usual topology and let $f : X \to X$ be the identity mapping. Since X isconnected, f is not subcontra-continuous. However, since X is T_1, f has the property that the inverse image of every (open) set is a union of closed sets.

4. APPLICATIONS TO COMPACT SPACES

In [1] Dontchev establishes that the image of an almost compact space under a contra-continuous, nearly continuous mapping is compact and that the contra-continuous image of a strongly S-closed space is compact. In this section, we strengthen both of these results by replacing contra-continuity with subcontra-continuity. The proofs mostly follow Dontchev's.

DEFINITION 6. Dontchev [1]. A space X is almost compact provided that every open cover of X has a finite subfamily the closures of whose members cover X.

THEOREM 7. The image of an almost compact space under a subcontra-continuous, nearly continuous mapping is compact.

PROOF. Let $f : X \to Y$ be subcontra-continuous and nearly continuous and assume that X is almost compact. Let $\mathcal{B}$ be an open base for the topology on Y for which $f^{-1}(V)$ closed in X for every $V \in \mathcal{B}$. Let $\mathcal{C}$ be an open cover of $f(X)$. For each $x \in X$, let $C_x \in \mathcal{C}$ such that $f(x) \in C_x$. Then let $V_x \in \mathcal{B}$ for which $f(x) \in V_x \subseteq C_x$. Now $f^{-1}(V_x)$ is closed and nearly open. It follows that $f^{-1}(V_x)$ is clopen and hence that $\{f^{-1}(V_x) : x \in X\}$ is a clopen cover of X. Since X is almost compact, there is a finite subfamily $\{f^{-1}(V_{x_i}) : i = 1, ..., n\}$ for which $X = \bigcup_{i=1}^{n} Cl(f^{-1}(V_{x_i})) = \bigcup_{i=1}^{n} f^{-1}(V_{x_i}) \subseteq \bigcup_{i=1}^{n} f^{-1}(C_{x_i})$. Thus we have that $f(X) \subseteq \bigcup_{i=1}^{n} C_{x_i}$ and therefore that $f(X)$ is compact. □

DEFINITION 7. Dontchev [1]. A space X is strongly S-closed provided that every closed cover of X has a finite subcover.

THEOREM 8. The subcontra-continuous image of a strongly S-closed space is compact.

PROOF. Let $f : X \to Y$ be subcontra-continuous and assume that X is strongly S-closed. Let $\mathcal{B}$ be an open base for the topology on Y for which $f^{-1}(V)$ is closed in X for every $V \in \mathcal{B}$. Let $\mathcal{C}$ be an open cover of $f(X)$. For each $x \in X$, let $C_x \in \mathcal{C}$ with $f(x) \in C_x$. Then let $V_x \in \mathcal{B}$ for which $f(x) \in V_x \subseteq C_x$. Since $\{f^{-1}(V_x) : x \in X\}$ is a closed cover of X and X is strongly S-closed, there is a finite subcover $\{f^{-1}(V_{x_i}) : i = 1,... ,n\}$ of X. Then we see that $f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(V_{x_i})\right) = \bigcup_{i=1}^{n} f(f^{-1}(V_{x_i})) \subseteq \bigcup_{i=1}^{n} V_{x_i} \subseteq \bigcup_{i=1}^{n} C_{x_i}$ and hence that $f(X)$ is compact. □

In [1] Dontchev also shows that the contra-continuous, β-continuous image of an S-closed space is compact. We extend this result by replacing contra-continuity with subcontra-continuity. The proof parallels that of Dontchev's.

DEFINITION 8. Mukherjee and Basu [11]. A space X is S-closed provided that every semi-open cover of X has a finite subfamily the closures of whose members covers X.

From Herrmann [12] , a space X is S-closed if and only if every regular closed cover of X has a finite subcover.

THEOREM 9. The subcontra-continuous, β-continuous image of an S-closed space is compact.

PROOF. Assume that $f : X \to Y$ is subcontra-continuous and β-continuous and that X is S-closed. Let $\mathcal{B}$ be an open base for the topology on Y for which $f^{-1}(V)$ is closed in X for every $V \in \mathcal{B}$. Let $\mathcal{C}$ be an open cover of $f(X)$. Then for each $x \in X$ there exists $C_x \in \mathcal{C}$ for which $f(x) \in C_x$. For each $x \in X$, let $V_x \in \mathcal{B}$ such that $f(x) \in V_x \subseteq C_x$. Since f is subcontra-continuous, $\{f^{-1}(V_x) : x \in X\}$ is a closed cover of X. The β-continuity of f implies that $f^{-1}(V_x) \subseteq Cl(Int(Cl(f^{-1}(V_x))))$ and therefore we see that $f^{-1}(V_x) = Cl(Int(f^{-1}(V_x)))$ or that

$f^{-1}(V_x)$ is regular closed. Since X is S-closed, the regular closed cover $\{f^{-1}(V_x) : x \in X\}$ has a finite subcover $\{f^{-1}(V_{x_i}) : i = 1,\dots, n\}$. Then we have $f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(V_{x_i})\right) \subseteq \bigcup_{i=1}^{n} V_{x_i} \subseteq \bigcup_{i=1}^{n} C_{x_i}$ and therefore $f(X)$ is compact. □

In the above proof we showed that, if $f : X \to Y$ is subcontra-continuous and β-continuous, then there exists an open base $\mathcal{B}$ for the topology on Y such that for every $V \in \mathcal{B}$, $f^{-1}(V)$ is regular closed and hence semi-open. Since unions of semi-open sets are semi-open (Arya and Bhamini [13]), it follows that inverse images of open sets are semi-open. Therefore we have the following theorem which strengthens the corresponding result for contra-continuous functions established by Dontchev [1].

THEOREM 10. Every subcontra-continuous, β-continuous function is semi-continuous.

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