Internat. J. Math. & Math. Sci.
VOL. 21 NO. 2 (1998) 239-248



# ON STRICT AND SIMPLE TYPE EXTENSIONS

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(Received August 14, 1996 and in revised form November 21, 1996)

**ABSTRACT.** Let $(Y,\tau)$ be an extension of a space $(X,\tau')$. $p \in Y$, let $\mathcal{O}_y^p = \{W \cap X : W \in \tau, p \in W\}$. For $U \in \tau'$, let $o(U) = \{p \in Y : U \in \mathcal{O}_y^p\}$. In 1964, Banaschweski introduced the strict extension $Y^{\#}$, and the simple extension $Y^{+}$ of $X$ (induced by $(Y,\tau)$) having base $\{o(U) : U \in \tau'\}$ and $\{U \cup \{p\} : p \in Y, \text{and } U \in \mathcal{O}_y^p\}$, respectively. The extensions $Y^{\#}$ and $Y^{+}$ have been extensively used since then. In this paper, the open filters $\mathcal{L}^p = \{W \in \tau' : W \supseteq \operatorname{int}_X \operatorname{cl}_X(U) \text{ for some } U \in \mathcal{O}_y^p\}$, and $\mathcal{U}^p = \{W \in \tau' : \operatorname{int}_X \operatorname{cl}_X(W) \in \mathcal{O}_y^p\} = \{W \in \tau' : \operatorname{int}_X \operatorname{cl}_X(W) \in \mathcal{L}^p\} = \cap\{\mathcal{U} : \mathcal{U}$ is an open ultrafilter on $X, \mathcal{O}_X^p \subset \mathcal{U}\}$ on $X$ are used to define some new topologies on $Y$. Some of these topologies produce nice extensions of $(X,\tau')$. We study some interrelationships of these extensions with $Y^{\#}$, and $Y^{+}$ respectively.

**KEY WORDS AND PHRASES:** Extension, simple extension, strict extension, H-closed, s-closed, almost realcompact, near compact.
**1980 AMS SUBJECT CLASSIFICATION CODES:** Primary 54D25.

## 1. INTRODUCTION

A topological space $Y$ is an *extension* of a space $X$ if $X$ is a dense subspace of $Y$. If $Y_1$ and $Y_2$ are two extensions of a space $X$, then $Y_2$ is said to be *projectively larger* than $Y_1$, written $Y_2 \geq Y_1$ (or $Y_1 \leq Y_2$), provided that there exists a continuous map $f : Y_2 \to Y_1$ such that $f|_X = i_X$, the identity map on $X$. Two extensions $Y_1$ and $Y_2$ of $X$ are called *equivalent* if $Y_1 \leq Y_2$ and $Y_2 \leq Y_1$. We shall identify two equivalent extensions of $X$. With this convention, the class $E(X)$ of all the Hausdorff extensions of a Hausdorff space $X$ is a set. Let $(Y,\tau) \in E(X)$ and let $p \in Y$. If $N_p$ is the open neighborhood filter of $p$ in $Y$, the set $\mathcal{O}_Y^p = \{N \cap X : N \in N_p\}$ (called the *trace* of $N_p$ on $X$) is an open filter on $X$. If $U$ is open in $X$, denote

$$o_Y(U) = \{p \in Y : U \in \mathcal{O}_Y^p\}.$$

In 1964 Banaschewski [1] introduced the extensions $Y^{\#}$ (resp. $Y^{+}$) the *strict extension* (resp. the *simple extension*) of $X$ induced by $Y$ satisfying $Y^{\#} \le Y \le Y^{+}$. The topology $\tau^{\#}$ on $Y^{\#}$ (resp. $\tau^{+}$ on $Y^{+}$) has for an open base the collection $\{o_Y(U): U \text{ open in } X\}$ (resp., the collection $\{U \cup \{p\}: p \in Y$, and $U \in \mathcal{O}_Y^p\}$). The extensions $Y^{\#}$, and $Y^{+}$ have been studied extensively and have proved extremely useful regarding some properties weaker than compactness, such as nearly compact, almost realcompact, feebly compact, $H$-closed, $s$-closed, etc.. In this paper we introduce new extensions $Y^{l}$, $Y^{u}$, $Y^{l^*}$, and $Y^{u^*}$, study some of their properties, and compare them with $Y$, $Y^{\#}$, and $Y^{+}$. All spaces under consideration are Hausdorff.

## 2. THE EXTENSIONS $Y^{l}$ AND $Y^{u}$.

In this section, we introduce several topologies on $Y$, and compare them with $\tau$. Some of these topologies yield interesting extensions of $(X, \tau')$.

**DEFINITION 2.1.** Let $(Y, \tau)$ be an extension of a space $(X, \tau')$. For $p \in Y$ define

$$\mathcal{U}^p = \{W: W \in \tau', \operatorname{int}_X \operatorname{cl}_X W \in \mathcal{O}_Y^p\}, \tag{2.1}$$

$$\mathcal{L}^p = \{W: W \in \tau', W \supseteq \operatorname{int}_X cl_X U \text{ for some } U \in \mathcal{O}_Y^p\}. \tag{2.2}$$

**LEMMA 2.1.**

(a) Both $\mathcal{U}^p$ and $\mathcal{L}^p$ are open filters on $X$ such that $\mathcal{L}^p \subseteq \mathcal{O}_Y^p \subseteq \mathcal{U}^p$.

(b) $\mathcal{U}^p = \{W: W \in \tau', \operatorname{int}_X \operatorname{cl}_X W \in \mathcal{L}^p\}$
$= \cap\{\mathcal{U}: \mathcal{U} \text{ is an open ultrafilter on } X, \mathcal{O}_Y^p \subset \mathcal{U}\}$

**PROOF.** We prove (b). Let $\mathcal{W} = \{W: W \in \tau', \operatorname{int}_X \operatorname{cl}_X W \in \mathcal{L}^p\}$. If $W \in \mathcal{W}$, then $W \in \tau'$ and $\operatorname{int}_X \operatorname{cl}_X W \supseteq \operatorname{int}_X \operatorname{cl}_X U$ for some $U \in \mathcal{O}_Y^p$. Therefore, $\operatorname{int}_X \operatorname{cl}_X W \in \mathcal{O}_Y^p$, whence $W \in \mathcal{U}^p$. Thus, $\mathcal{W} \subseteq \mathcal{U}^p$. To prove the reverse inequality, let $W \in \mathcal{U}^p$. Then $\operatorname{int}_X \operatorname{cl}_X W \in \mathcal{O}_Y^p$. Since $\operatorname{int}_X \operatorname{cl}_X W \supseteq \operatorname{int}_X \operatorname{cl}_X(\operatorname{int}_X \operatorname{cl}_X W)$ it follows that $\operatorname{int}_X \operatorname{cl}_X W \in \mathcal{L}^p$. Hence $W \in \mathcal{W}$. This proves the first equality in (b). The second equality follows from [9], completing the proof of the lemma.

**REMARK 2.1.** Since $\mathcal{O}_Y^p = \mathcal{O}_Y^p\# = \mathcal{O}_Y^p +$[9,10,11], it follows that each one of $Y$, $Y^{+}$ and $Y^{\#}$ yield the same $\mathcal{L}^p$ (resp., $\mathcal{U}^p$) for all $p \in Y$. Moreover, if $Z \in E(X)$ has the same underlying set as $Y$, and is such that $Y^{\#} \le Z \le Y^{+}$, then $Y$ and $Z$ induce the same $\mathcal{L}^p$ (resp., $\mathcal{U}^p$) for all $p \in Y$. Also, if $p \ne q$ are distinct elements of $Y$ then $\mathcal{L}^p \ne \mathcal{L}^q$ and $\mathcal{U}^p \ne \mathcal{U}^q$. Obviously, if $U \in \mathcal{O}_Y^p$, then $\operatorname{int}_X \operatorname{cl}_X(U) \in \mathcal{L}^p$. Moreover, $U \in \mathcal{U}^p$ if and only if $\operatorname{int}_X \operatorname{cl}_X(U) \in \mathcal{U}^p$.

**DEFINITION 2.2.** Let $(Y, \tau)$ be an extension of $(X, \tau')$. For $G \in \tau'$, define

$$o_l(G) = G \cup \{p: p \in Y \setminus X, G \in \mathcal{L}^p\} \tag{2.3}$$

$$o_u(G) = G \cup \{p: p \in Y \setminus X, G \in \mathcal{U}^p\} \tag{2.4}$$

$$a_l(G) = \{p \in Y: G \in \mathcal{L}^p\} \tag{2.5}$$

$$a_u(G) = \{p \in Y: G \in \mathcal{U}^p\} \tag{2.6}$$

The proof of the Propositions 2.1, and 2.2 is straightforward.

**PROPOSITION 2.1.** Let $(Y,\tau)$ be an extension of $(X,\tau')$. Then for all $U,V \in \tau'$

(a) $o_l(\varnothing) = \varnothing, o_l(X) = Y$,

(b) $o_l(U) \cap X = U$,

(c) $o_l(U \cap V) = o_l(U) \cap o_l(V)$,

(d) The family $\{o_l(G): G \in \tau'\}$ is an open base for a Hausdorff topology $\tau_l$ on $Y$ and $(Y,\tau_l)$ is an extension of $X$.

**PROPOSITION 2.2.** Let $(Y,\tau)$ be an extension of $(X,\tau')$. Then for all $U,V \in \tau'$,

(a) $o_u(\varnothing) = \varnothing$ and $o_u(X) = Y$,

(b) $o_u(U) \cap X = U$,

(c) $o_u(U \cap V) = o_u(U) \cap o_u(V)$,

(d) The family $\{o_u(G): G \in \tau'\}$ is an open base for a Hausdorff topology $\tau_u$ on $Y$ and $(Y,\tau_u)$ is an extension of $X$.

**PROPOSITION 2.3.** Let $(Y,\tau)$ be an extension of $(X,\tau')$. Then for all $U,V \in \tau'$

(a) $a_l(\varnothing) = \varnothing, a_l(X) = Y$,

(b) $a_l(U) \cap X \subseteq U$,

(c) $a_l(U \cap V) = a_l(U) \cap a_l(V)$,

(d) $a_l(U) = \cup\{W: W \in \tau \text{ and } \operatorname{int}_X \operatorname{cl}_X(W \cap X) \subseteq U\}$

(e) The family $\{a_l(G): G \in \tau'\}$ is an open base for a coarser Hausdorff topology $\tau_{al}$ on $Y$, $X$ is dense in $(Y,\tau_{al})$, but $(Y,\tau_{al})$ may not be an extension of $X$.

**PROOF.** We prove (d). The rest is straightforward. Let $p \in a_l(U)$. Then $U \in \mathcal{L}^p$. Therefore, $U \supseteq \operatorname{int}_X \operatorname{cl}_X V$ for some $V \in \mathcal{O}_Y^p$. Therefore, there exists $W \in \tau$ such that $p \in W$ and $W \cap X = V$. It follows that $\operatorname{int}_X \operatorname{cl}_X(W \cap X) \subseteq U$. Conversely, if $W \in \tau$ is such that $\operatorname{int}_X \operatorname{cl}_X(W \cap X) \subseteq U$ and $p \in W$, then $W \cap X \in \mathcal{O}_Y^p$. So, $\operatorname{int}_X \operatorname{cl}_X(W \cap X) \in \mathcal{L}^p$. This implies that $U \in \mathcal{L}^p$ and hence $p \in a_l(U)$. The proof of the proposition is now complete.

**PROPOSITION 2.4.** Let $(Y,\tau)$ be an extension of $(X,\tau')$. Then for all $U,V \in \tau'$,

(a) $a_u(\varnothing) = \varnothing$ and $a_u(X) = Y$,

(b) $a_u(U) \cap X = \operatorname{int}_X \operatorname{cl}_X(U)$,

(c) $a_u(U \cap V) = a_u(U) \cap a_u(V)$,

(d) $a_u(U) = \cup\{W : W \in \tau \text{ and } W \cap X \subseteq \operatorname{int}_X \operatorname{cl}_X(U)\}$

(e) The family $\{a_u(G) : G \in \tau'\}$ is an open base for a coarser Hausdorff topology $\tau_{au}$ on $Y$, $X$ is dense in $(Y, \tau_{au})$, but $(Y, \tau_{au})$ may not be an extension of $X$.

**PROOF.** We prove (d). The rest is straightforward. Let $p \in a_u(U)$. Then $U \in \mathcal{U}^p$. Therefore, $\operatorname{int}_X \operatorname{cl}_X U \in \mathcal{O}_Y^p$. It follows that there exists $W \in \tau$ such that $p \in W$ and $W \cap X \subseteq \operatorname{int}_X \operatorname{cl}_X U$. Conversely, if $W \in \tau$ is such that $W \cap X \subseteq \operatorname{int}_X \operatorname{cl}_X U$ and $p \in W$, then $W \cap X \in \mathcal{O}_Y^p$. So, $\operatorname{int}_X \operatorname{cl}_X U \in \mathcal{O}_Y^p$. Therefore, $U \in \mathcal{U}^p$ and $p \in a_u(U)$.

**DEFINITION 2.3.** The spaces $(Y, \tau_l)$, $(Y, \tau_u)$, $(Y, \tau_{al})$, and $(Y, \tau_{au})$ described in propositions 2.1-2.4 will, henceforth, be denoted by $Y^l$, $Y^u$, $Y^{al}$, and $Y^{au}$ respectively. If $A \subseteq Y$, then $\operatorname{int}_{Y^l}(A)$ (resp. $\operatorname{cl}_{Y^l}(A)$) will be denoted by $\operatorname{int}_l(A)$ (resp., $\operatorname{cl}_l(A)$). Likewise, $\operatorname{int}_u(A), \operatorname{cl}_u(A), \operatorname{int}_{al}(A), \operatorname{cl}_{al}(A), \operatorname{int}_{au}(A)$, and $\operatorname{cl}_{au}(A)$ are defined in an analogous manner.

**LEMMA 2.2.** If $U \in \tau'$, then

(a) $a_l(U) \subseteq o_l(U) \subseteq o_Y(U) \subseteq o_u(U) \subseteq o_u(\operatorname{int}_X \operatorname{cl}_X U) = a_u(U) = a_u(\operatorname{int}_X \operatorname{cl}_X U)$,

(b) $a_l(U) \setminus X = o_l(U) \setminus X$, and $a_u(U) \setminus X = o_u(U) \setminus X$

(c) $o_l(\operatorname{int}_X \operatorname{cl}_X U) \setminus X = o_u(U) \setminus X$, and

(d) if $U$ is regular open (*i.e.* $U = \operatorname{int}_X \operatorname{cl}_X U$), then $a_u(U) = a_l(U)$, and the equality holds in (a).

**PROOF.** Part (a): We show that $o_u(\operatorname{int}_X \operatorname{cl}_X U) = a_u(U)$, the rest being straightforward. Certainly, $o_u(\operatorname{int}_X \operatorname{cl}_X U) \cap X = \operatorname{int}_X \operatorname{cl}_X U = a_u(U) \cap X$. Let $p \in o_u(\operatorname{int}_X \operatorname{cl}_X U) \setminus X$. Then $\operatorname{int}_X \operatorname{cl}_X U \in \mathcal{U}^p$. Therefore, $U \in \mathcal{U}^p$, and $p \in a_u(U) \setminus X$. Conversely, let $p \in a_u(U) \setminus X$. Then, $U \in \mathcal{U}^p$. So, $p \in o_u(U) \setminus X \subseteq o_u(\operatorname{int}_X \operatorname{cl}_X U) \setminus X$. The above arguments prove (a).

To prove (c), let $q \in o_l(\operatorname{int}_X \operatorname{cl}_X G) \setminus X$. Then, $\operatorname{int}_X \operatorname{cl}_X G \in \mathcal{L}^q$ whence, $G \in \mathcal{U}^q$. Therefore, $q \in o_u(G) \setminus X$. Thus, $o_l(\operatorname{int}_X \operatorname{cl}_X G) \setminus X \subseteq o_u(G) \setminus X$. To prove the reverse inequality, let $q \in o_u(G) \setminus X$. Then, $G \in \mathcal{U}^q$, whence $\operatorname{int}_X \operatorname{cl}_X G \in \mathcal{L}^q$. Therefore, $q \in o_l(\operatorname{int}_X \operatorname{cl}_X G) \setminus X$ and $o_u(G) \setminus X \subseteq o_l(\operatorname{int}_X \operatorname{cl}_X G) \setminus X$. Hence, $o_l(\operatorname{int}_X \operatorname{cl}_X G) \setminus X = o_u(G) \setminus X$. The rest of the lemma is straightforward.

Given a space $(X, \tau')$, the family $\{\operatorname{int}_X \operatorname{cl}_X U : U \in \tau'\}$ forms an open base for a coarser Hausdorff topology $\tau'_s$ on $X$. The space $X_s = (X, \tau'_s)$ is called the *semiregularization* of $X$. A space $(X, \tau')$ is called *semiregular* if $(X, \tau') = X_s$

**THEOREM 2.1.** If $X$ is semiregular, and $(Y, \tau)$ (not necessarily semiregular) is an extension of $X$, then $Y^l$ is an extension of $X$ such that $Y^l \leq Y$.

**PROOF.** If $X$ is semiregular, then $o_l(U) = a_l(U)$ for all $U \in \tau'$. Hence, $Y^l$ is an extension of $X$ such that $Y^l = Y^{al} \leq Y$.

**THEOREM 2.2.** The spaces $Y^{al}$ and $Y^{au}$ are homeomorphic.

**PROOF.** For all $U \in \tau'$, $a_l(\operatorname{int}_X \operatorname{cl}_X U) = o_u(\operatorname{int}_X \operatorname{cl}_X U) = a_u(U)$ implies that $\tau_{au} \subseteq \tau_{al}$. Also, if $G \in \tau'$ and $p \in a_l(G)$, then $G \supseteq \operatorname{int}_X \operatorname{cl}_X(U)$ for some $U \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$. Now, if $q \in a_u(U)$, then $\operatorname{int}_X \operatorname{cl}_X U \in \mathcal{L}^q$ which implies that $G \in \mathcal{L}^q$, or $q \in a_l(G)$. Therefore, $p \in a_u(U) \subseteq a_l(G)$. Hence, $\tau_{al} \subseteq \tau_{au}$. This proves the theorem.

**LEMMA 2.3.** Let $(Y,\tau)$ be an extension of $(X,\tau')$. Then, for all $G \in \tau'$ the following are true.

(a) $\operatorname{cl}_{al}(G) \subseteq \operatorname{cl}_l(G) = \operatorname{cl}_l(\operatorname{int}_X \operatorname{cl}_X(G))$,

(b) $\operatorname{cl}_u(G) = \operatorname{cl}_{au}(G) = \operatorname{cl}_u(\operatorname{int}_X \operatorname{cl}_X G))$,

(c) $\operatorname{cl}_u(G) = \operatorname{cl}_l(G)$,

(d) $\operatorname{cl}_Y(G) = \operatorname{cl}_{au}(G) = \operatorname{cl}_{al}(\operatorname{int}_X \operatorname{cl}_X(G))$, and

(e) $\operatorname{cl}_u(o_u(G)) = \operatorname{cl}_{au}(a_u(\operatorname{int}_X \operatorname{cl}_X(G)))$

**PROOF.** Part (a): Let $p \in \operatorname{cl}_{al}(G)$, and let $o_l(U)$ be a basic open neighborhood of $p$ in $Y^l$. If $p \in o_l(U) \cap X$, then $p \in U \subseteq \operatorname{int}_X \operatorname{cl}_X U \in \mathcal{L}^p$. Therefore, $a_l(\operatorname{int}_X \operatorname{cl}_X U)$ is an open neighborhood of $p$ in $Y^{al}$. Consequently, $a_l(\operatorname{int}_X \operatorname{cl}_X U) \cap G \neq \varnothing$. By Proposition (2.7) (b), $\operatorname{int}_X \operatorname{cl}_X U \cap G \neq \varnothing$. Hence $U \cap G \neq \varnothing$. This in turn implies that $o_l(U) \cap G \neq \varnothing$, and $p \in \operatorname{cl}_l(G)$. If $p \in o_l(U) \setminus X$, then $U \in \mathcal{L}^p$. Now, $a_l(U)$ is an open neighborhood of $p$ in $Y^{al}$. Consequently, $a_l(U) \cap G \neq \varnothing$. Therefore, $o_l(U) \cap G \neq \varnothing$ whence $p \in \operatorname{cl}_l(G)$.

Part (b): Let $p \in \operatorname{cl}_{au}(G)$, and let $o_u(U)$ be a basic open neighborhood of $p$ in $Y^u$. Since $o_u(U) \subseteq a_u(U), a_u(U)$ is an open neighborhood of $p$ in $Y^{au}$. Hence, $a_u(U) \cap G \neq \varnothing$. Therefore, $\operatorname{int}_X \operatorname{cl}_X U \cap G \neq \varnothing$, whence $U \cap G \neq \varnothing$. Consequently, $o_u(U) \cap G \neq \varnothing$. Therefore, $p \in cl_u(G)$. Therefore, $\operatorname{cl}_{au}(G) \subseteq \operatorname{cl}_u(G)$. Conversely, let $p \in \operatorname{cl}_u(G)$, and let $a_u(U)$ be a basic open neighborhood of $p$ in $Y^{au}$. If $p \in a_u(U) \cap X = \operatorname{int}_X \operatorname{cl}_X U$, then $o_u(\operatorname{int}_X \operatorname{cl}_X U)$ is an open neighborhood of $p$ in $Y^u$. Therefore, $a_u(U) \cap X = \operatorname{int}_X \operatorname{cl}_X U \cap G = o_u(\operatorname{int}_X \operatorname{cl}_X U) \cap G \neq \varnothing$. Hence, $p \in \operatorname{cl}_{au}(G)$. Now, if $p \in a_u(U) \setminus X$, then $U \in \mathcal{U}^p$ and $o_u(U)$ is an open neighborhood of of $p$ in $Y^u$. Therefore, $o_u(U) \cap G \neq \varnothing$. Consequently, $a_u(U) \cap G \neq \varnothing$, and $p \in \operatorname{cl}_{au}(G)$. Therefore, $\operatorname{cl}_u(G) \subseteq \operatorname{cl}_{au}(G)$. Hence, $\operatorname{cl}_u(G) \subseteq \operatorname{cl}_{au}(G)$. The other half of (b) is straightforward.

The proof of (c) is straightforward.

Part (d): Let $p \in \operatorname{cl}_{au}(G)$, and let $W$ be an open neighborhood of $p$ in $Y$. Then, $W \cap X \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$ shows that $o_u(W \cap X)$ is an open neighborhood of $p$ in $Y^{au}$. Therefore, $a_u(W \cap X) \neq \varnothing$. This shows that $W \cap G \neq \varnothing$, whence $p \in \operatorname{cl}_Y(G)$. Conversely, let $p \in \operatorname{cl}_Y(G)$, and let $a_u(U)$ be a basic open neighborhood of $p$ in $Y^{au}$. Then, $U \in \mathcal{U}^p$. So, $o_Y(\operatorname{int}_X \operatorname{cl}_X U)$ is an open neighborhood of $p$ in $Y$ such that $o_Y(\operatorname{int}_X \operatorname{cl}_X U) \cap G \neq \varnothing$. This implies that $a_u(U) \cap G \neq \varnothing$. Hence, $p \in \operatorname{cl}_{au}(G)$. The rest follows from (c).

**THEOREM 2.3.** The spaces $Y^l \setminus X, Y^{al} \setminus X$, and $Y^u \setminus X$ are pairwise homeomorphic.

**PROOF.** To prove the continuity of the identity map $i:Y^u \setminus X \to Y^l \setminus X$, let $o_l(G)\setminus X$ be a basic open neighborhood of $p$ in $Y^l \setminus X$. Then, $G \in \mathcal{L}^p$. Hence $G \supseteq \operatorname{int}_X \operatorname{cl}_X U$ for some $U \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$. Therefore, $o_u(U)\setminus X$ is an open neighborhood of $p$ in $Y^u$ such that $o_u(U)\setminus X \subseteq o_l(G)\setminus X$. To prove that the identity map $i:Y^l \setminus X \to Y^u \setminus X$ is continuous, let $o_u(G)\setminus X$ be a basic open neighborhood of $p$ in $Y^u \setminus X$. Then $o_l(\operatorname{int}_X \operatorname{cl}_X G)\setminus X$ is an open neighborhood of $p$ in $Y^l \setminus X$ such that $o_l(\operatorname{int}_X \operatorname{cl}_X G)\setminus X = o_u(G)\setminus X$. Hence, the spaces $Y^l \setminus X$, and $Y^u \setminus X$ are homeomorphic. The rest of the theorem follows directly from Lemma 2.2.

Let $Z_1$ and $Z_2$ be spaces. A map $f:Z_1 \to Z_2$ is called $\theta$ *-continuous* [3] if for every $p \in Z_1$ and for every open neighborhood $V$ of $f(p)$ in $Z_2$, there exists an open neighborhood $U$ of $p$ in $Z_1$ such that $f(\operatorname{cl}_{Z_1} U) \subseteq \operatorname{cl}_{Z_2}(V)$. $f$ is called *perfect* if $f$ is a closed map (not necessarily continuous) such that $f^{\leftarrow}(z)$ is compact in $Z_1$ for every $z \in Z_2$. Also, $f$ is called *irreducible* if $f$ is closed and there is no proper closed subset $K$ of $Z_1$ for which $f(K) = Z_2$. Two extensions $Z_1$, and $Z_2$ of a space $X$ are called $\theta$ *- equivalent* if there exists a $\theta$ - homeomorphism $f$ from $Z_1$ onto $Z_2$ such that $f|_X = i_X$, the identity map on $X$.

The next theorem depicts some of the several interrelationships between the spaces $Y$, $Y^\#$, $Y^l$, $Y^u$, and $Y^{al}$.

**THEOREM 2.4.** Let $(Y,\tau)$ be an extension of a space $(X,\tau')$. The following statements are true.

(a) The identity map $i:Y^{al} \to Y$ is perfect, irreducible and $\theta$ - continuous.

(b) The identity map $i:Y^{au} \to Y^u$ is perfect, irreducible and $\theta$ - continuous.

(c) The identity map $i:Y^{al} \to Y^\#$ is $\theta$ - continuous.

(d) The identity map $i:Y^\# \to Y^l$ is $\theta$ - continuous.

(e) The identity map $i:Y^\# \to Y^u$ is $\theta$ - continuous.

(f) The identity map $i:Y^l \to Y^\#$ is $\theta$ - continuous.

(g) The identity map $i:Y^u \to Y^\#$ is $\theta$ - continuous.

(h) The identity map $i:Y^l \to Y^u$ is $\theta$ - continuous.

(i) The identity map $i:Y^u \to Y^l$ is $\theta$ - continuous.

(j) The identity map $i:Y^l \to Y$ is $\theta$ - continuous.

(k) The identity map $i:Y^u \to Y$ is $\theta$ - continuous.

(l) The identity map $i:Y^\# \to Y^{al}$ is $\theta$ - continuous.

**PROOF.** Below, we outline the proofs of some parts of the theorem. The rest of the proofs are analogous.

Part (a) Since $\tau_{al} \subseteq \tau, i:Y \to Y^{al}$ is continuous. Hence, $i:Y \to Y^{al}$ is irreducible and perfect. To prove the $\theta$ - continuity of $i:Y^{al} \to Y$, let $V$ be an open neighborhood of $p$ in $Y$. Then $V \cap X \in \mathcal{O}_Y^p$ and $\operatorname{int}_X \operatorname{cl}_X (V \cap X) \in \mathcal{L}^p$. Therefore, $a_l(\operatorname{int}_X \operatorname{cl}_X (V \cap X))$ is an open neighborhood of $p$ in $Y^{al}$ such that

$\mathrm{cl}_{al}(a_l(\mathrm{int}_X \mathrm{cl}_X(V \cap X)) = \mathrm{cl}_Y(a_l(\mathrm{int}_X \mathrm{cl}_X(V \cap X)) = \mathrm{cl}_Y[a_l(\mathrm{int}_X \mathrm{cl}_X(V \cap X)) \cap X]$
$= \mathrm{cl}_Y(\mathrm{int}_X \mathrm{cl}_X(V \cap X)) \subseteq \mathrm{cl}_Y(V)$. Hence $i: Y^{al} \to Y$ is $\theta$-continuous.

Part (b): For all $G \in \tau', a_u(G) = o_u(\mathrm{int}_X \mathrm{cl}_X G) \in \tau_u$ shows that $i: Y^u \to Y^{au}$ is continuous. Therefore, $i: Y^{au} \to Y$ is irreducible and perfect. Let $o_u(G)$ be a basic open neighborhood of $p$ in $Y^u$. Since $o_u(G) \subseteq a_u(G), a_u(G)$ is an open neighborhood of $p$ in $Y^{au}$ such that $\mathrm{cl}_{au}(a_u(G)) = \mathrm{cl}_u(o_u(G))$, establishing the $\theta$-continuity of $i: Y^{au} \to Y^u$.

Part (c): To prove the $\theta$-continuity of $i: Y^{al} \to Y^{\#}$, let $p \in Y$ and let $o_Y(G), G \in \tau'$ be a basic open neighborhood of $p$ in $Y^{\#}$. Then, $G \in \mathcal{O}_Y^p \subseteq \mathcal{U}^p$ implies that $a_l(\mathrm{int}_X \mathrm{cl}_X G)$ is an open neighborhood of $p$ in $Y^{al}$ such that $\mathrm{cl}_{al}(a_l(\mathrm{int}_X \mathrm{cl}_X G)) \subseteq \mathrm{cl}_Y\#(o_Y(G))$

Part (d): Let $o_l(G)$ be a basic open neighborhood of $p$ in $Y^l$. Then, $o_Y(G)$ is an open neighborhood of $p$ in $Y^{\#}$ such that $\mathrm{cl}_Y\#(o_Y(G)) \subseteq \mathrm{cl}_l(o_l(G))$, establishing the $\theta$-continuity of $i: Y^{\#} \to Y^l$.

Part (h): Let $o_u(G)$ be a basic open neighborhood of $p$ in $Y^u$. Then, $o_l(\mathrm{int}_X \mathrm{cl}_X G)$ is an open neighborhood of $p$ in $Y^l$ satisfying $\mathrm{cl}_l(o_l(\mathrm{int}_X \mathrm{cl}_X G)) \subseteq \mathrm{cl}_u(o_u(G))$

Part (l): Let $p \in Y$ and let $a_l(G), G \in \tau'$ be a basic open neighborhood of $p$ in $Y^{al}$. Then, $G \supseteq \mathrm{int}_X \mathrm{cl}_X U$ for some $U \in \mathcal{O}_Y^p$. So, $p \in o_Y(U)$. Now, $\mathrm{cl}_Y(o_Y(U)) = \mathrm{cl}_Y(o_Y(U) \cap X) = \mathrm{cl}_Y(U) \subseteq \mathrm{cl}_{al}(U) \subseteq \mathrm{cl}_{al}(a_l(G))$.

We now summarize the results proved above in the following theorem.

**THEOREM 2.5.** The spaces $Y$, $Y^{\#}$, $Y^l$, $Y^u$, and $Y^{al}$, are pairwise $\theta$-homeomorphic. The spaces $Y^l$, and $Y^u$ are $\theta$-equivalent extensions of $X$ with homeomorphic remainders.

It is well known that spaces $Y$ and $Z$ are $\theta$-homeomorphic if and only if their semiregularizations are homeomorphic. [11] Hence, we have the following corollary.

**COROLLARY 2.1.** Let $(Y,\tau)$ be an extension of a space $(X,\tau)$. Then, the spaces $Y_s, Y_s^{\#}, Y_s^l, Y_s^u$, and $Y_s^{al}$ are pairwise homeomorphic. Moreover, $Y_s, Y_s^l$, and $Y_s^u$ are equivalent extensions of $X_s$.

## 3. THE EXTENSIONS $Y^{l*}$, AND $Y^{u*}$.

In this section, we define extensions $Y^{l*}$, and $Y^{u*}$, analogous to the simple extension $Y^+$ of $(X,\tau')$ induced by an extension $(Y,\tau)$ of $X$. The spaces $Y^{l*}$, $Y^{al}$, $Y^{u*}$, and $Y^{au*}$ all have the same underlying set as the set $Y$. An open base for the topology $\tau_{l*}$ on $Y^{l*}$ (respectively, $\tau_{al*}$ on $Y^{al*}$) is the family $\tau' \cup \{G \cup \{p\} : p \in Y \setminus X, G \in \mathcal{L}^p\}$ (respectively, $\tau' \cup \{G \cup \{p\} : G \in \mathcal{L}^p\}$). An open base for the topology $\tau_{u*}$ on $Y^{u*}$ (respectively, $\tau_{au*}$ on $Y^{au*}$) is the family $\tau' \cup \{G \cup \{p\} : p \in Y \setminus X, G \in \mathcal{U}^p\}$ (respectively, $\tau' \cup \{G \cup \{p\} : G \in \mathcal{U}^p\}$). For any $A \subset Y, \mathrm{cl}_{l*}(A)$ will denote the closure of $A$ in $Y^{l*}$, with analogous notations in other cases. The proofs of the following statements are straightforward, and we omit the details. Obviously, the spaces $Y^{l*} \setminus X, Y^{u*} \setminus X, Y^{al*} \setminus X$, and $Y^{au*} \setminus X$ are all discrete.

**THEOREM 3.1.** The spaces $Y^{l*}$, and $Y^{u*}$ are extensions of $(X,\tau')$ such that $Y^{u*} \geq Y^+ \geq Y^{l*}$. The set $X$ is dense in the spaces $Y^{al*}$, and $Y^{au*}$. But, $Y^{al*}$ and $Y^{au*}$ may not be extensions of $X$.

**LEMMA 3.1.** For each $G \in \tau'$, $\mathrm{cl}_{l*}(o_l(G)) = \mathrm{cl}_l(o_l(G)$, and $\mathrm{cl}_{u*}(o_u(G)) = \mathrm{cl}_u(o_u(G))$.

**THEOREM 3.2.** Each one of the identity maps $i: Y^+ \to Y^{u*}$, and $i: Y^{l*} \to Y^+$ is $\theta$-continuous.

**THEOREM 3.3.** The spaces $Y^+$, $Y^{l*}$, $Y^{u*}$, $Y^{al*}$, and $Y^{au*}$ are $\theta$-homeomorphic. Moreover, $Y^+$, $Y^{l*}$, and $Y^{u*}$ are $\theta$-equivalent extensions of $X$ with homeomorphic remainders.

**COROLLARY 3.1.** If $(Y,\tau)$ is an extension of a space $(X,\tau')$, then the spaces $Y_s^+, Y_s^{l*}, Y_s^{u*}, Y_s^{al*}$, and $Y_s^{au*}$ are homeomorphic in pairs. Moreover, the spaces $Y_s^+, Y_s^{l*}$, and $Y_s^{u*}$ are equivalent extensions of $X_s$

**REMARKS 3.1.** (a) If **P** is any property of topological spaces which is preserved under $\theta$-continuous surjections, and if $(Y,\tau)$ is a **P**-extension of $(X,\tau')$, then $Y^l$, $Y^u$, $Y^{l*}$, and $Y^{u*}$ are also **P**-extensions of $X$.

(b) The extensions $Y^l$, $Y^u$, $Y^{l*}$, and $Y^{u*}$ introduced above are, in general, all distinct from $Y$, $Y^\#$, and $Y^+$. It would be interesting to find a characterization of spaces $Y$ for which $Y^\# = Y^l$. A space $Z$ is called $H$-closed if it is closed in every Hausdorff space in which it is embedded [see 11 for more details]. The Katetov (respectively, Fomin) extension of a space $(X,\tau')$ is the space $\kappa X$ (respectively, $\sigma X$) whose underlying set is the set $X \cup \{p : p \text{ is a free open ultrafilter on } X\}$, and whose topology has for an open base the family $\tau' \cup \{U \cup \{p\} : U \in p, \text{ and } p \in \kappa X \setminus X\}$ (respectively, the family $\{o_{\kappa X}(U) : U \in \tau'\}$). The spaces $\kappa X$, and $\sigma X$ are $H$-closed extensions of $X$ such that $(\sigma X)^+ = \kappa X$, and $(\kappa X)^\# = \sigma X$ [3, 6, 11]. In general $(\sigma X)^l \neq \sigma X, (\kappa X)^u \neq \kappa X, (\sigma X)^{u*} \neq \sigma X$, and $(\kappa X)^{l*} \neq \kappa X$. Analogous remarks apply to the Banaschewski-Fomin-Shanin extension $\mu X$ [13] of a Hausdorff space $X$

(c) A space $Z$ is called *compact like*, or *nearly compact* if every regular open cover of $Z$ is reducible to a finite subcover. A space $X$ has a compactlike extension if and only if $X_s$ is Tychonoff [14]. Compactlike extensions (=*near* compactifications) of Hausdorff almost completely regular spaces $X$ (whence, $X_s$ is Tychonoff) have been constructed in [2] via EF-Proximities. For a Hausdorff space $X$ whose semiregularization $X_s$ is Tychonoff, a maximal compactlike extension $BX$ of $X$, satisfying $(BX)_s = \beta X_s$, is constructed in [14]. If $(X,\tau')$ is any Hausdorff almost complètely regular space, and if $(Y,\tau)$ is any near compactification of $(X,\tau')$, then so are $Y^l$, $Y^u$, $Y^{l*}$, and $Y^{u*}$.

(d) A space $Z$ is called almost real compact if every open ultrafilter on $Z$ with countable closed intersection property in $Z$ converges in $Z$[4]. A space $Z$ is almost realcompact if and only if $Z_s$ is almost realcompact [12]. Almost realcompactifications of a Hausdorff space have been constructed (among others) in [7], and [12]. If $(X,\tau')$ is any Hausdorff space, and if $(Y,\tau)$ is any almost realcompactification of $(X,\tau')$, then so are $Y^l$, $Y^u$, $Y^{l*}$, and $Y^{u*}$.

(e) A Hausdorff space $Z$ is called extremally disconnected if for each open subset $U$ of $Z$, $cl_Z(U)$ is open. A space $Z$ is extremally disconnected if and only if each dense subspace of $Z$ [respectively, if and only if $Z_s$] is extremally disconnected [see 11 for more details]. A Hausdorff space $Z$ is called *s-closed* if it is $H$-closed and extremally disconnected [8]. A Hausdorff space $Z$ is $s$-closed if and only if $Z_s$ is $s$-closed. It is shown in [8] that every extremally disconnected space $X$ admits an $s$-closed extension ,viz.

$\kappa X$ ; moreover, an extension $Y$ of $X$ is $s$-closed if and only if $X$ is $C^*$-embedded in $Y$. If $(X,\tau')$ is any extremally disconnected Hausdorff space, and if $(Y,\tau)$ is any $s$-closed extension of $(X,\tau')$, then so are $Y^l$, $Y^u$, $Y^{l*}$, and $Y^{u*}$.

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