WEAK REGULARITY OF PROBABILITY MEASURES

DALE SIEGEL

Kingsborough Community College
Mathematics/Computer Science Department
2001 Oriental Boulevard
Brooklyn, New York 11235

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ABSTRACT. This paper examines smoothness attributes of probability measures on lattices which indicate regularity, and then discusses weaker forms of regularity; specifically, weakly regular and vaguely regular. They are obtained from commonly used outer measures, and we study them mainly for the case of $M(\mathcal{L})$ or for those components of $M(\mathcal{L})$ with added smoothness prerequisites. This is a generalization of many concepts presented in my earlier paper (see [1]).

KEY WORDS AND PHRASES: Lattice regular, σ -smooth, and outer measures. Weakly and vaguely regular measures. Normal and complement generated lattices.

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1. INTRODUCTION

Let X be an arbitrary set and $\mathcal{L}$ a lattice of subsets of X . $A(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$ and $M(\mathcal{L})$ those finitely additive measures on $A(\mathcal{L})$. $M_\sigma(\mathcal{L})$ denotes those elements of $M(\mathcal{L})$ that are σ -smooth on $\mathcal{L}$; while $M_R(\mathcal{L})$ denotes those elements of $M(\mathcal{L})$ that are $\mathcal{L}$ -regular. To each $\mu \in M(\mathcal{L})$ we will associate a finitely subadditive outer measure μ' on $P(X)$, and to $\mu \in M_\sigma(\mathcal{L})$ is associated an outer measure μ'' . The relationships between μ , μ' , and μ'' on $\mathcal{L}$ and $\mathcal{L}'$ (the complementary lattice) are investigated. This leads to a consideration of weak notions of regularity, which can be expressed in terms of μ' and μ'' . In this respect the normal lattices are particularly important since for such lattices regularity of μ coincides with weak regularity. We show that if $\mu \in N(\mathcal{L})$, those $\mu \in M(\mathcal{L})$ such that for $L_n \downarrow L$, $L_n, L \in \mathcal{L}$, $\mu(L) = \inf_n \mu(L_n)$ and if $\mathcal{L}$ is complement generated then μ is weakly regular. Combining these results gives conditions for certain measures to be regular. We adhere to standard lattice and measure terminology which will be used throughout the paper (see e.g. [2-6]) and review some of this in section two for the reader's convenience.

2. DEFINITIONS AND NOTATIONS

Let X be an abstract set. Let $\mathcal{L}$ be a lattice of subsets of X . We assume throughout that $\emptyset$ and X are in $\mathcal{L}$. If $A \subset X$, then we will denote the complement of A by A' (i.e. $A' = X - A$). If $\mathcal{L}$ is a lattice of subsets of X , then $\mathcal{L}' = \{L' | L \in \mathcal{L}\}$ is the complementary lattice of $\mathcal{L}$.

LATTICE TERMINOLOGY

DEFINITION 2.1. Let $\mathcal{L}$ be a lattice of subsets of X . We say that:

1. $\mathcal{L}$ is a δ -lattice if it is closed under countable intersections; $\delta(\mathcal{L})$ is the lattice of countable intersections of sets of $\mathcal{L}$.
2. $\mathcal{L}$ is disjunctive if and only if $x \in X$, $L \in \mathcal{L}$, and $x \notin L$ imply there exists $A \in \mathcal{L}$ such that $x \in A$ and $A \cap L = \emptyset$.

3. $\mathcal{L}$ is complement generated if $L \in \mathcal{L}$ implies $L = \bigcap_{n=1}^{\infty} L'_n$, where $L_n \in \mathcal{L}$.
4. $\mathcal{L}$ is compact if and only if $X = \bigcup_{\alpha} L'_\alpha$, $L_\alpha \in \mathcal{L}$, implies there exists a finite number of L'_α that cover X .
5. $\mathcal{L}$ is countably compact if and only if $X = \bigcup_{i=1}^{\infty} L'_i$, $L_i \in \mathcal{L}$, implies there exists a finite number of the L'_i that cover X .
6. $\mathcal{L}$ is countably paracompact if, for every sequence $\{L_n\}$ in $\mathcal{L}$ such that $L_n \downarrow \emptyset$, there exists a sequence $\{\tilde{L}_n\}$ in $\mathcal{L}$ such that $L_n \subset \tilde{L}'_n$ and $\tilde{L}_n \downarrow \emptyset$.
7. $\mathcal{L}$ is normal if and only if $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$ imply there exists $C, D \in \mathcal{L}$ such that $A \subset C'$, $B \subset D'$, and $C' \cap D' = \emptyset$.

MEASURE TERMINOLOGY

Let $\mathcal{L}$ be a lattice of subsets of X . $M(\mathcal{L})$ will denote the set of finite-valued, bounded, finitely additive measures on $A(\mathcal{L})$. We may clearly assume throughout that all measures are non-negative.

DEFINITION 2.2.

1. A measure $\mu \in M(\mathcal{L})$ is said to be σ -smooth on $\mathcal{L}$ if $L_n \in \mathcal{L}$ and $L_n \downarrow \emptyset$ imply $\mu(L_n) \rightarrow 0$.
2. A measure $\mu \in M(\mathcal{L})$ is said to be σ -smooth on $A(\mathcal{L})$ if $A_n \in A(\mathcal{L})$ and $A_n \downarrow \emptyset$ imply $\mu(A_n) \rightarrow 0$.
3. A measure $\mu \in M(\mathcal{L})$ is said to be $\mathcal{L}$ -regular if, for any $A \in A(\mathcal{L})$,

$$\mu(A) = \sup\{\mu(L) : L \subset A, L \in \mathcal{L}\}.$$

NOTATION 2.3. If $\mathcal{L}$ is a lattice of subsets of X , then we will denote by:

- $M_\sigma(\mathcal{L})$ = the set of σ -smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$
- $M^\sigma(\mathcal{L})$ = the set of σ -smooth measures on $A(\mathcal{L})$ of $M(\mathcal{L})$
- $M_R(\mathcal{L})$ = the set of $\mathcal{L}$ -regular measures of $M(\mathcal{L})$
- $M_R^\sigma(\mathcal{L})$ = the set of $\mathcal{L}$ -regular measures of $M^\sigma(\mathcal{L})$

DEFINITION 2.4.

1. Let $\mu \in M(\mathcal{L})$. Then $\mu \in N(\mathcal{L})$ if $L_n \in \mathcal{L}$ and $\bigcap_{n=1}^{\infty} L_n = L \in \mathcal{L}$ (in particular, if $\mathcal{L}$ is δ), $L_n \downarrow$, imply $\mu(L) = \inf \mu(L_n)$.
2. If $\mu \in M(\mathcal{L})$, then the support of μ is $S(\mu) = \bigcap \{L \in \mathcal{L} \mid \mu(L) = \mu(X)\}$.

REMARK 2.5. Listed below are a few basic important facts that will be used throughout the paper (see [7,8] for further details):

1. $M_R^\sigma(\mathcal{L}) = M_R(\mathcal{L}) \cap M_\sigma(\mathcal{L})$
2. $M_\sigma(\mathcal{L}) \supset N(\mathcal{L}) \supset M^\sigma(\mathcal{L})$
3. If $\mu \in M(\mathcal{L})$, then there exists $\nu \in M_R(\mathcal{L})$ such that $\mu \leq \nu(\mathcal{L})$ (i.e. $\mu(L) \leq \nu(L)$, all $L \in \mathcal{L}$) and $\mu(X) = \nu(X)$.

3. REGULAR PROBABILITY MEASURES

Discussion of $\mathcal{L}$ -regular measures ($\mu \in M_R(\mathcal{L})$) takes place in this section. Conditions for regularity and various resulting properties are examined.

THEOREM 3.1. Let $\mathcal{L}$ be a lattice of subsets of X . Then $\mu \in M_R(\mathcal{L})$ if and only if $\mu \in M(\mathcal{L})$ and $\mu(A) = \inf\{\mu(L') : A \subset L', L \in \mathcal{L}\}$, $A \in A(\mathcal{L})$.

PROOF. 1. Suppose $\mu \in M_R(\mathcal{L})$. Then $\mu(A') = \sup\{\mu(L) : L \subset A', L \in \mathcal{L}\}$. Hence

$$\begin{aligned}
\mu(A) &= \mu(X) - \mu(A') = \mu(X) - \sup\{\mu(L) : L \subset A', L \in \mathcal{L}\} \\
&= \mu(X) - \sup\{\mu(L) : L' \supset A, L \in \mathcal{L}\} \\
&= \mu(X) - \sup\{\mu(X) - \mu(L') : L' \supset A, L \in \mathcal{L}\} \\
&= \mu(X) - \mu(X) + \sup\{-\mu(L') : L' \supset A, L \in \mathcal{L}\}.
\end{aligned}$$

Therefore $\mu(A) = \inf\{\mu(L') : A \subset L', L \in \mathcal{L}\}$.

2. Reverse of 1. is sufficient proof.

THEOREM 3.2. Let $\mathcal{L}$ be a lattice of subsets of X . Suppose $\mu \in M(\mathcal{L})$ and $\mu(L') = \sup\{\mu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$. Then $\mu \in M_R(\mathcal{L})$.

PROOF. Suppose $\mu \in M(\mathcal{L})$ and $\mu(L') = \sup\{\mu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$. This implies $\mu(L) = \inf\{\mu(\tilde{L}') : L \subset \tilde{L}', \tilde{L}' \in \mathcal{L}\}$, by Theorem 3.1. Let $A \in A(\mathcal{L})$. By definition, $A = \bigcup_{i=1}^n L_i \cap \tilde{L}'_i$, where $L_i, \tilde{L}'_i \in \mathcal{L}$ and disjoint. Consider $L \cap \tilde{L}'$; $L, \tilde{L}' \in \mathcal{L}$. Since every $L \in \mathcal{L}$ is $\mathcal{L}'$ -outer regular with respect to μ , there exists $\tilde{L}' \supset L, \tilde{L}' \in \mathcal{L}$, such that $\mu(L) + \epsilon > \mu(\tilde{L}')$, $\epsilon > 0$. Then $\tilde{L}' \cap \tilde{L}' \supset L \cap \tilde{L}'$ and $\tilde{L}' \cup \tilde{L}' \supset L \cup \tilde{L}'$.

$$\mu(L \cap \tilde{L}') = \mu(L) + \mu(\tilde{L}') - \mu(L \cup \tilde{L}') \geq \mu(\tilde{L}') - \epsilon + \mu(\tilde{L}') - \mu(\tilde{L}' \cup \tilde{L}') = \mu(\tilde{L}' \cap \tilde{L}') - \epsilon.$$

Consequently, $\mu(L \cap \tilde{L}') + \epsilon \geq \mu(\tilde{L}' \cap \tilde{L}')$. Therefore, $L \cap \tilde{L}'$ is $\mathcal{L}'$ -outer regular with respect to μ .

Now, in general, $A = \bigcup_{i=1}^n (L_i \cap \tilde{L}'_i)$, where $L_i, \tilde{L}'_i$ disjoint and $\epsilon > 0$. There exists $\tilde{L}'_i \supset L_i \cap \tilde{L}'_i$ such that $\mu(L_i \cap \tilde{L}'_i) + \frac{\epsilon}{2^n} > \mu(\tilde{L}'_i)$. Then $\bigcup_{i=1}^n \tilde{L}'_i \supset \bigcup_{i=1}^n (L_i \cap \tilde{L}'_i) = A$ and $\bigcup_{i=1}^n \tilde{L}'_i \in \mathcal{L}'$.

$$\mu(A) = \mu\left(\bigcup_{i=1}^n (L_i \cap \tilde{L}'_i)\right) = \sum_{i=1}^n \mu(L_i \cap \tilde{L}'_i) \geq \sum_{i=1}^n \mu(\tilde{L}'_i) - \sum_{i=1}^n \frac{\epsilon}{2^n} \geq \mu\left(\bigcup_{i=1}^n \tilde{L}'_i\right) - \epsilon.$$

Hence $\mu(A) = \inf\{\mu(L') : A \subset L', L \in \mathcal{L}\}$. Therefore $\mu \in M_R(\mathcal{L})$, by 3.1.

THEOREM 3.3 Let $\mu_1 \in M_R(\mathcal{L})$, $\mu_2 \in M(\mathcal{L})$, $\mu_1 \leq \mu_2(\mathcal{L})$, and $\mu_1(X) = \mu_2(X)$. Then $\mu_1 = \mu_2$.

PROOF. Suppose $\mu_1 \leq \mu_2(\mathcal{L})$ and let $L \subset \tilde{L}', L, \tilde{L}' \in \mathcal{L}$. This implies $\mu_2 \leq \mu_1(\mathcal{L}')$ and $\mu_2(L) \leq \mu_2(\tilde{L}') \leq \mu_1(\tilde{L}')$. If $\mu_2(L) \leq \mu_1(\tilde{L}')$, for any $\tilde{L}' \supset L$, then $\mu_2(L) \leq \inf\{\mu_1(\tilde{L}') : L \subset \tilde{L}'\} = \mu_1(L)$, since $\mu \in M_R(\mathcal{L})$. Hence $\mu_2 \leq \mu_1(\mathcal{L})$ and, consequently, $\mu_1 = \mu_2(\mathcal{L})$. Therefore $\mu_1 = \mu_2$ since $\mu_1(X) = \mu_2(X)$.

THEOREM 3.4. Let $\mathcal{L}$ be a lattice of subsets of X . Suppose $\mu \in M_R(\mathcal{L})$ and $\mu \in M_\sigma(\mathcal{L})$. Then $\mu \in M^\sigma(\mathcal{L})$.

PROOF. Given $\mu \in M_R(\mathcal{L})$ and $\mu \in M_\sigma(\mathcal{L})$. Let $\{A_n\}$ be in $A(\mathcal{L})$ and $A_n \downarrow \emptyset$. Then there exists $L_n \subset A_n$, $L_n \in \mathcal{L}$, and $\mu(A_n) - \frac{\epsilon}{2^n} < \mu(L_n)$, since $\mu \in M_R(\mathcal{L})$. Now, $L_1, L_1 \cap L_2, L_1 \cap L_2 \cap L_3, \dots$ are in $\mathcal{L}$ and $\downarrow \emptyset$. So $\mu(L_1 \cap L_2) \leq \mu(A_1 \cap A_2) = \mu(A_2) \leq \mu(L_1 \cap L_2) + \frac{\epsilon}{2} + \frac{\epsilon}{4}$. By induction, $\mu\left(\bigcap_{i=1}^n A_i\right) \leq \mu\left(\bigcap_{i=1}^n L_i\right) + \sum_{i=1}^n \frac{\epsilon}{2^i}$ for all n . Consequently, we may assume $L_n \downarrow \emptyset$ and $\mu(A_n) < \mu(L_n) + \epsilon$, all n . Then $\lim_{n \rightarrow \infty} \mu(A_n) \leq \lim_{n \rightarrow \infty} \mu(L_n) + \epsilon$; and $\lim_{n \rightarrow \infty} \mu(L_n) = 0$ since $\mu \in M_\sigma(\mathcal{L})$. This implies $\lim_n \mu(A_n) \leq \epsilon$ and $\epsilon > 0$. Hence $\lim_n \mu(A_n) = 0$. Therefore $\mu \in M^\sigma(\mathcal{L})$, since μ is countably additive on $A(\mathcal{L})$.

THEOREM 3.5. Let $\mu \leq \nu(\mathcal{L})$, where $\mu \in M(\mathcal{L})$, $\nu \in M_R(\mathcal{L})$, and $\mu(X) = \nu(X)$. If $\mathcal{L}$ is normal, then $\nu(L') = \sup\{\mu(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\}$.

PROOF. Since $\nu \in M_R(\mathcal{L})$, $\nu(L') = \sup\{\nu(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\}$. This implies $\nu(L') - \epsilon < \nu(\tilde{L})$, $\epsilon > 0$, for some $\tilde{L} \in \mathcal{L}$ where $\tilde{L} \subset L$. By normality, $\tilde{L} \subset A' \subset B \subset L'$, where $A, B \in \mathcal{L}$.

Then $\nu(L') - \epsilon < \nu(\tilde{L}) \leq \nu(A') \leq \mu(A') \leq \mu(B) \leq \nu(B) \leq \nu(L)$. Hence $\nu(L') = \sup\{\mu(B) : B \subset L', B \in \mathcal{L}\}$, and $\mu(B) \sim \mu(\tilde{L})$ by an ϵ -argument. Therefore $\nu(L') = \sup\{\mu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$.

THEOREM 3.6. Suppose $\mu \in M_R(\mathcal{L})$ and $\lambda \in M_R(\mathcal{L}')$ such that $\mu \leq \lambda(\mathcal{L}')$. Then $\mathcal{L}$ is normal if and only if $\mu(L') = \sup\{\lambda(A) : A \subset L', A \in \mathcal{L}\}$.

PROOF.

1. $\mu \leq \lambda(\mathcal{L}')$ implies $\lambda \leq \mu(\mathcal{L})$, by regularity. Therefore, if $\mathcal{L}$ is normal, then $\mu(L') = \sup\{\lambda(A) : A \subset L', A \in \mathcal{L}\}$ by 3.5.
2. Suppose $\mu(L') = \sup\{\lambda(A) : A \subset L', A \in \mathcal{L}\}$. Let $\mu_1, \mu_2 \in M_R(\mathcal{L})$ such that $\mu \leq \mu_1(\mathcal{L})$ and $\mu \leq \mu_2(\mathcal{L})$. Then $\mu_1 \leq \mu \leq \lambda(\mathcal{L}')$ and $\mu_2 \leq \mu \leq \lambda(\mathcal{L}')$. This implies $\mu_1(L') = \mu_2(L') = \sup\{\lambda(A) : A \subset L', A \in \mathcal{L}\}$. Hence $\mu_1 = \mu_2$. Therefore, $\mathcal{L}$ is normal.

THEOREM 3.7. Suppose $\mathcal{L}$ is normal and complement generated. Then $\mu \in N(\mathcal{L})$ implies $\mu \in M_R^g(\mathcal{L})$.

PROOF. Since $\mathcal{L}$ is complement generated, there exists $L, L_n \in \mathcal{L}$ such that $L = \bigcap_{n=1}^{\infty} L'_n$, where $L_n \downarrow$. By normality, $L \subset A'_n \subset B_n \subset L'_n$, where $A_n, B_n \in \mathcal{L}$, and we may assume that $A_n \downarrow, B_n \downarrow$. Then $L = \bigcap_n B_n = \bigcap_n L'_n$. Now let $\mu \in N(\mathcal{L})$. This implies $\mu(L) = \inf_n \mu(B_n) = \inf_n \mu(A'_n)$. Hence $\mu \in M_R(\mathcal{L})$ by 3.1, and $N(\mathcal{L}) \subset M_\sigma(\mathcal{L})$ by 2.5. Therefore $\mu \in M_R^g(\mathcal{L})$.

4. OUTER MEASURES

In this section we consider $\mu \in M(\mathcal{L})$, and associate with it certain "outer measures" μ' and μ'' . In general, they differ from the customary induced "outer measures" μ^* and μ_* . We seek to investigate the interplay of these outer measures on the lattice $\mathcal{L}$ and, conversely, the effect of $\mathcal{L}$ on them.

DEFINITION 4.1. Let $\mu \in M(\mathcal{L})$ such that $\mu \geq 0$ and let E be a subset of X .

1. $\mu'(E) = \inf\{\mu(L') : E \subset L', L \in \mathcal{L}\}$ is a finitely-subadditive outer measure.
2. $\mu''(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(L'_n) : E \subset \bigcup_{n=1}^{\infty} L'_n, L_n \in \mathcal{L}\right\}$ is a countably-subadditive outer measure.
3. $\mu^*(E) = \inf\{\mu(A) : E \subset A, A \in \mathcal{A}(\mathcal{L})\}$ is a finitely-subadditive outer measure.
4. $\mu_*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : E \subset \bigcup_{i=1}^{\infty} A_i, A_i \in \mathcal{A}(\mathcal{L})\right\}$ is a countably-subadditive outer measure.

DEFINITION 4.2.

1. Suppose ν is an outer measure and let E be a subset of X . Then $E \in \mathcal{S}_\nu$, the set of ν -measurable sets, if $\nu(A) = \nu(A \cap E) + \nu(A \cap E')$ for all $A \subset X$.
2. ν is said to be a regular outer measure if, for $A, E \subset X$, there exists $E \in \mathcal{S}_\nu$ such that $A \subset E$ and $\nu(A) = \nu(E)$.

PROPERTY 4.3. Proofs will be omitted.

1. If $\mathcal{L}$ is countably compact and $\mu \in M(\mathcal{L})$, then $\mu' = \mu''(\mathcal{L})$.
2. If $\mu \in N(\mathcal{L})$, then $\mu' = \mu''(\mathcal{L}')$.
3. $\mu \in M_\sigma(\mathcal{L})$ and $\mu' = \mu''(\mathcal{L}')$, where μ'' is regular, imply $\mu \in N(\mathcal{L})$.
4. If $\mu \in N(\mathcal{L})$ and $\mathcal{L}$ is δ , then $\mu' = \mu''(\mathcal{L})$.
5. Suppose $\mu \in N(\mathcal{L})$, $\mathcal{L}$ is δ , and $\mathcal{L} \subset \mathcal{S}_{\mu''}$. Then $\mu \in M_R^g(\mathcal{L})$.

THEOREM 4.4. Let $\mu \in M_\sigma(\mathcal{L})$. Then

- (a) $\mu(X) = \mu''(X)$,
- (b) $\mu \leq \mu'' \leq \mu'(\mathcal{L})$,
- (c) $\mu'' \leq \mu = \mu'(\mathcal{L}')$.

PROOF. (a) Clearly $\mu''(X) \leq \mu(X)$. If $\mu''(X) < \mu(X)$, then there exists $L'_i \in \mathcal{L}'$, $i = 1, 2, \dots$, such that $X = \bigcup_{i=1}^{\infty} L'_i$ and $\sum_{i=1}^{\infty} \mu(L'_i) < \mu(X)$. But $\sum_{i=1}^{\infty} \mu(L'_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(L'_i) \geq \lim_{n \rightarrow \infty} \mu\left(\bigcup_{i=1}^n L'_i\right)$. Also

$\bigcup_1^n L'_i \uparrow X$ and $\bigcup_1^n L'_i \in \mathcal{L}'$. This implies that $\lim_n \mu\left(\bigcup_1^n L'_i\right) = \mu(X)$ since $\mu \in M_\sigma(\mathcal{L})$. Therefore $\mu(X) = \mu''(X)$.

(b) Suppose there exists $L \in \mathcal{L}$ such that $\mu(L) > \mu''(L)$. Then $\mu''(X) \leq \mu''(L) = \mu''(L') < \mu(L) + \mu''(L')$. Then $\mu''(X) \leq \mu''(L) + \mu''(L') < \mu(L) + \mu''(L')$, but $\mu'' \leq \mu(\mathcal{L})$. This implies $\mu''(X) < \mu(L) + \mu(L') = \mu(X)$, which contradicts (a). Hence $\mu \leq \mu''(\mathcal{L})$; and $\mu'' \leq \mu'$ everywhere clearly. Thus $\mu'' \leq \mu'(\mathcal{L})$. Therefore $\mu \leq \mu'' \leq \mu'(\mathcal{L})$.

(c) Clearly $\mu'' \leq \mu'(\mathcal{L}')$ and, by definition, $\mu = \mu'(\mathcal{L}')$. Therefore $\mu'' \leq \mu = \mu'(\mathcal{L}')$.

THEOREM 4.5. Suppose ν is a finite, regular, finitely-subadditive outer measure defined on $P(X)$, the set of all subsets of X . Then $E \in \mathcal{S}_\nu$ if and only if $\nu(X) = \nu(E) + \nu(E')$.

PROOF. 1. Suppose ν is a finitely-subadditive regular outer measure and $E \in \mathcal{S}_\nu$. Then $\nu(E) + \nu(E') = \nu(X)$, clearly.

2. Suppose ν is a finitely-subadditive regular outer measure and $\nu(X) = \nu(E) + \nu(E')$. Let $B \in \mathcal{S}_\nu$. Then, by regularity, there exists a set $F \subset X$ such that $F \subset B$ and $\nu(F) = \nu(B)$. Then, since $B \in \mathcal{S}_\nu$, $\nu(E) = \nu(E \cap B) + \nu(E \cap B')$ and $\nu(E') = \nu(E' \cap B) + \nu(E' \cap B')$. So

$$\begin{aligned} \nu(X) &= \nu(E) + \nu(E') = \nu(E \cap B) + \nu(E \cap B') + \nu(E' \cap B) + \nu(E' \cap B') \\ &\geq \nu(B) + \nu(B') = \nu(X), \end{aligned}$$

since $B \in \mathcal{S}_\nu$. Also, $\nu(B \cap E) + \nu(B \cap E') + \nu(B' \cap E) + \nu(B' \cap E') = \nu(B) + \nu(B')$ since all $\nu(X) = \text{finite measure}$. Now subtract from the equation above $\nu(B' \cap E) + \nu(B' \cap E') \geq \nu(B')$, which is true by the finite subadditivity of ν . Then $\nu(B \cap E) + \nu(B \cap E') \leq \nu(B)$. Also, $F \cap E \subset B \cap E$ and $F \cap E' \subset B \cap E'$. This implies

$$\nu(F \cap E) + \nu(F \cap E') \leq \nu(B \cap E) + \nu(B \cap E') \leq \nu(B) = \nu(F).$$

Hence $\nu(F) = \nu(F \cap E) + \nu(F \cap E')$. Therefore $E \in \mathcal{S}_\nu$.

THEOREM 4.6. Suppose $\mu \leq \nu(\mathcal{L})$, where $\mu \in M(\mathcal{L})$ and $\nu \in M_R(\mathcal{L})$. Then:

(a) $\mu \leq \nu = \nu' \leq \mu'(\mathcal{L})$

(b) if $\mathcal{L}$ is normal, then $\mu' = \nu'(\mathcal{L})$.

PROOF. (a) Since $\nu \in M_R(\mathcal{L})$, $\nu(E) = \nu'(E) = \inf\{\nu(L') : E \subset L', L' \in \mathcal{L}\}$. Also, $\mu \leq \nu(\mathcal{L})$ implies $\nu \leq \mu(\mathcal{L}')$, which implies $\nu' \leq \mu'(\mathcal{L})$ and $\nu' \leq \mu'(\mathcal{L}')$. Therefore $\mu \leq \nu = \nu' \leq \mu'(\mathcal{L})$.

(b) Let $L \in \mathcal{L}$. Then, by normality,

$$\begin{aligned} \nu'(L) &= \nu(L) = \nu(X) - \nu(L') = \nu(X) - \sup\{\nu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\} \\ &= \nu(X) - \sup\{\mu(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\} \\ &= \inf\{\mu(\tilde{L}') : \tilde{L}' \supset L\} = \mu'(L). \end{aligned}$$

Therefore $\mu' = \nu'(\mathcal{L})$.

5. WEAKER NOTIONS OF REGULARITY

Previously we have considered some properties related to $\mu \in M_R(\mathcal{L})$. We now want to consider weaker notions of regularity, and see when they might coincide with regularity; and, in general, to investigate their properties and interplay with the underlying lattice.

DEFINITION 5.1. Let $L \in \mathcal{L}$, where $\mathcal{L}$ is a lattice of subsets of X .

1. A measure $\mu \in M(\mathcal{L})$ is said to be *weakly regular* if $\mu(L') = \sup\{\mu'(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$.

2. A measure $\mu \in M_\sigma(\mathcal{L})$ is said to be *vaguely regular* if $\mu(L') = \sup\{\mu''(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$.

NOTATION 5.2.

$M_W(\mathcal{L})$ = the set of weakly regular measures of $M(\mathcal{L})$

$M_V(\mathcal{L})$ = the set of vaguely regular measures of $M_\sigma(\mathcal{L})$

LEMMA 5.3. $M_R^q(\mathcal{L}) \subset M_V(\mathcal{L}) \subset M_W(\mathcal{L}) \cap M_\sigma(\mathcal{L})$

REMARK 5.4. If $\mu' = \mu''(\mathcal{L})$, then $M_V(\mathcal{L}) = M_W(\mathcal{L}) \cap M_\sigma(\mathcal{L})$. This occurs if:

- (a) $\mathcal{L}$ is countably compact,
- (b) $\mu \in N(\mathcal{L})$ and $\mathcal{L}$ is δ ,
- (c) $\mathcal{L}$ is normal and complement generated,
- (d) $\mathcal{L}$ is δ -normal.

THEOREM 5.5. Suppose $\mu \leq \nu(\mathcal{L})$, where $\mu \in M_W(\mathcal{L})$ and $\nu \in M_R(\mathcal{L})$. Then $\mu' = \nu(\mathcal{L})$ implies $\mu = \nu$.

PROOF. Let $M_W(\mathcal{L}) \ni \mu \leq \nu \in M_R(\mathcal{L})$ and suppose $\mu' = \nu(\mathcal{L})$. Then $\mu \leq \nu = \nu' \leq \mu'(\mathcal{L})$ by 4.6. Now, $\mu \in M_W(\mathcal{L})$ implies $\mu(L') = \sup\{\mu'(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\}$ and $\nu \in M_R(\mathcal{L})$ implies $\nu(L') = \sup\{\nu(\tilde{L}) : \tilde{L} \subset L', L, \tilde{L} \in \mathcal{L}\}$. Then, since $\mu' = \nu(\mathcal{L})$, $\mu(L') = \nu(L')$, which implies $\mu = \nu(\mathcal{L}')$. Therefore $\mu = \nu$, since $\mu(X) = \nu(X)$.

THEOREM 5.6. Suppose $\mathcal{L}$ is complement generated. If $\mu \in N(\mathcal{L})$ and μ'' is a regular outer-measure, then $\mu \in M_V(\mathcal{L}) \subset M_W(\mathcal{L}) \cap M_\sigma(\mathcal{L})$.

PROOF. Suppose $\mathcal{L}$ is complement generated and $\mu \in N(\mathcal{L})$. Then $\mu \in M_\sigma(\mathcal{L})$ by 2.5; and $\mu = \mu' = \mu''(\mathcal{L}')$, by 4.3 and 4.4. Now let $L \in \mathcal{L}$. Then, since $\mathcal{L}$ is complement generated, $L = \bigcap_{n=1}^{\infty} L'_n$, $L_n \in \mathcal{L}$, $L'_n \downarrow$. By the regularity of μ'' and the fact that $L' \subset \bigcap_{n=1}^{\infty} L'_n$, we have $\mu''(L') = \lim_n \mu''(L'_n)$. But $\mu = \mu' = \mu''(\mathcal{L}')$ since $\mu \in N(\mathcal{L})$. Thus $\mu(L') = \sup\{\mu''(\tilde{L}) : \tilde{L} \subset L', \tilde{L} \in \mathcal{L}\}$. Hence $\mu \in M_V(\mathcal{L})$. Therefore, by 5.3, $\mu \in M_V(\mathcal{L}) \subset M_W(\mathcal{L}) \cap M_\sigma(\mathcal{L})$.

THEOREM 5.7. Suppose $\mathcal{L}$ is normal and $\mu \in M_W(\mathcal{L})$. Then $\mu \in M_R(\mathcal{L})$.

PROOF. Suppose $\mathcal{L}$ is normal and $\mu \in M_W(\mathcal{L})$. Let $\mu \leq \nu(\mathcal{L})$, where $\nu \in M_R(\mathcal{L})$. Then, using 4.6,

$$\begin{aligned}\nu(L') &= \sup\{\nu(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\} = \sup\{\nu'(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\} \\ &= \sup\{\mu'(\tilde{L}) : \tilde{L} \subset L'; L, \tilde{L} \in \mathcal{L}\} = \mu(L') \quad \text{since } \mu \in M_W(\mathcal{L}).\end{aligned}$$

So $\mu = \nu(\mathcal{L}')$, which implies $\mu = \nu$ since $\mu(X) = \nu(X)$. Therefore $\mu \in M_R(\mathcal{L})$.

REMARK 5.8. We saw in Theorem 5.7 that if $\mathcal{L}$ is normal, then $M_W(\mathcal{L}) = M_R(\mathcal{L})$. However, the converse is not true. For example, let $\mathcal{L} = \{\emptyset, X, A, B, A \cup B\}$, where $A, B \subset X (A, B \neq \emptyset)$ such that $A \cap B = \emptyset$ and $A \cup B \neq X$. Here $\mathcal{L}$ is clearly not normal, but $M_W(\mathcal{L}) = M_R(\mathcal{L})$.

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