ABSTRACT. A fixed point theorem is proved in a Banach space $E$ which has uniformly normal structure for asymptotically regular mapping $T$ satisfying:

for each $x, y$ in the domain and for $n = 1, 2, \cdots$,

\[ \|T^n x - T^n y\| \leq a_n \|x - y\| + b_n (\|x - T^n x\| + \|y - T^n y\|) + c_n (\|x - T^n y\| + \|y - T^n x\|), \]

where $a_n, b_n, c_n$ are nonnegative constants satisfying certain conditions. This result generalizes a fixed point theorem of Górnicki [1].

KEY WORDS AND PHRASES: Uniformly normal structure, asymptotic regularity, fixed point.


1. INTRODUCTION

Let $E$ be a Banach space and $K$ a nonempty, bounded, closed and convex subset of $E$. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. Browder [2], Gohde [3] and Kirk [4] proved independently that if $E$ is uniformly convex, then $T$ always has a fixed point in $K$ (see also Goebel [5]). Now, it is important (cf. [4]) that if one assumes $T$ to be Lipschitzian with Lipschitz constant $k > 1$, then $T$ need not have a fixed point, even if $E$ is a Hilbert space and $k$ is an arbitrary near 1. However, there are classes of transformations which lie between the nonexpansive transformation and those with Lipschitz constant $k > 1$ for which fixed point theorems do exist, in particular, the asymptotically nonexpansive mappings (cf. [6]) form such a class. These are mappings $T : K \to K$ having the property that $T^n$ has Lipschitz constant $k_n$ with $k_n \to 1$ as $n \to \infty$.

In this paper, we obtain a fixed point theorem for the class of mappings whose $n$th iterate $T^n$ satisfy:

\[ \|T^n x - T^n y\| \leq a_n \|x - y\| + b_n (\|x - T^n x\| + \|y - T^n y\|) + c_n (\|x - T^n y\| + \|y - T^n x\|), \]

for each $x, y \in K$ and $n = 1, 2, \cdots$, where $a_n, b_n, c_n$ are nonnegative constants such that there exists an integer $n_0$ such that $b_n + c_n < 1$ for all $n \geq n_0$. This class of mappings are more general than nonexpansive mappings. Also by taking $b_n = c_n = 0$ it will be seen that this class of mappings are more general than asymptotically nonexpansive mappings. Our result improves and extends the results of Górnicki [1] and others.
2. **PRELIMINARIES**

The concept of uniformly normal structure is due to Gillespie and Williams [7]. A Banach space \( E \) has uniformly normal structure if

\[
N(E) = \sup \{ r_K(K) : K \subset E \text{ is convex and } \text{diam } K = 1 \} < 1,
\]

where

\[
r_K(K) = \inf \{ \sup \{ \| x - y \| : y \in K \} : x \in K \}.
\]

It was proved in [8], [9] that \( N(E) \leq 1 - \delta_E(1) \); thus \( \varepsilon_0(E) < 1 \) implies uniformly normal structure, where \( \delta_E(\cdot) \) is the modulus of convexity of \( E \) and \( \varepsilon_0(E) \) is the characteristic of convexity of \( E \). Yu [10] proved that if \( E \) is a uniformly smooth space, then \( E \) has a uniformly normal structure. Also, in [11] it was proved that uniformly normal structure does not necessarily imply that the space has good geometric properties.

The following lemma is needed to prove our main result:

**LEMMA 1** [12]. Let \( K \) be a nonempty closed convex subset of a Banach space \( E \) and let \( \{ n_i \} \) be an increasing sequence of natural numbers. Assume that \( T : K \rightarrow K \) is an asymptotically regular mapping such that for some \( n \in \mathbb{N} \), \( T^n \) is continuous. If

\[
\lim_{i \to \infty} \| z - T^{n_i}x \| = 0
\]

for some \( x \in K \) and \( z \in K \), then \( Tz = z \).

3. **MAIN RESULTS**

Now we state and prove our main result:

**THEOREM 1.** Let \( K \) be a nonempty closed convex subset of a Banach space \( E \) which has uniformly normal structure, i.e. \( N(E) < 1 \). Let \( T : K \rightarrow K \) be an asymptotically regular mapping which holds the inequality (1) such that \( (\alpha + \beta) \cdot \gamma \cdot N(E) < 1 \), where

\[
\alpha = \liminf_{n \to \infty} \frac{a_n + c_n}{1 - c_n},
\]

\[
\beta = \liminf_{n \to \infty} \frac{b_n}{1 - c_n},
\]

and

\[
\gamma = \liminf_{n \to \infty} \frac{a_n + c_n}{1 - c_n - b_n}.
\]

Suppose that there is a \( z_0 \) in \( K \) for which \( \{ T^n z_0 \} \) is bounded. Then \( T \) has a fixed point in \( K \).

**PROOF.** Let \( \{ n_i \} \) be a sequence of natural numbers such that

\[
\alpha = \liminf_{n \to \infty} \frac{a_n + c_n}{1 - c_n} = \lim_{i \to \infty} \frac{a_n + c_n}{1 - c_n},
\]

\[
\beta = \liminf_{n \to \infty} \frac{b_n}{1 - c_n} = \lim_{i \to \infty} \frac{b_n}{1 - c_n},
\]

and

\[
\gamma = \liminf_{n \to \infty} \frac{a_n + c_n}{1 - c_n - b_n} = \lim_{i \to \infty} \frac{a_n + c_n}{1 - c_n - b_n}.
\]

Since \( \{ T^n z_0 \} \) is bounded (and hence \( \{ T^n z \} \) is bounded for any \( z \) in \( K \)), by Lemma 1, we can inductively construct a sequence \( \{ z_n \} \) such that \( z_n \) is the unique asymptotic center of the sequence \( \{ T^n z_{n-1} \} \), with respect to the functional

\[
\lim \sup_{n \to \infty} \| x - T^n z_{n-1} \|
\]
over \( x \) in \( K \). Now for each \( m \geq 1 \), we set

\[
D_m = \lim_{i \to \infty} \|z_m - T^m z_m\|
\]

and

\[
r_m = \lim_{i \to \infty} \|z_{m+1} - T^m z_m\|.
\]

Using (1), we have

\[
\|T^nx - T^n y\| \leq \|T^nx - T^{n+y}\| + \|T^{n+y} y - T^n y\|
\]

\[
\leq a_n \|x - T^n y\| + b_n \left( \|x - T^n x\| + \|T^n y - T^n x\| \right)
\]

\[
+ c_n \left( \|T^n y - T^n x\| + \|T^{n+y} y - T^n y\| \right)
\]

implies

\[
\|T^nx - T^n y\| \leq \frac{a_n + c_n}{1 - c_n} \cdot \|x - T^n y\| + \frac{b_n}{1 - c_n} \cdot \|x - T^n x\|
\]

\[
+ \frac{1 + b_n + c_n}{1 - c_n} \cdot \|T^n y - T^n x\|.
\]  

(2)

By inequality (2), the result of Casini and Maluta [13], and the asymptotic regularity of \( T \), we have

\[
r_m \leq N(E) \cdot \limsup_{i \to \infty} \|T^n z_m - T^n z_m\| \cdot \|n, n_j \geq s\)
\]

\[
\leq N(E) \cdot \limsup_{i \to \infty} \left( \limsup_{j \to \infty} \|T^n z_m - T^n z_m\| \right)
\]

\[
\leq N(E) \cdot \limsup_{i \to \infty} \left( \limsup_{j \to \infty} \left\{ \frac{a_n + c_n}{1 - c_n} \cdot \|x_m - T^n z_m\| + \frac{b_n}{1 - c_n} \cdot \|x_m - T^n z_m\|
\right.
\]

\[
+ \frac{1 + b_n + c_n}{1 - c_n} \cdot \sum_{i=0}^{n-1} \|T^{n+y} z_m - T^{n+y} z_m\| \right\}
\]

and so

\[
r_m \leq (\alpha + \beta) \cdot N(E) \cdot D_m, \quad m = 0, 1, \cdots,
\]  

(3)

where \( N(E) \) is the normal structure coefficient of \( E \). Moreover, for \( i > 1 \), we have

\[
\|T^n z_m - z_m\| \leq \limsup_{j \to \infty} \|T^n z_m - T^n z_m\| \leq \limsup_{j \to \infty} \left\{ \frac{a_n + c_n}{1 - c_n} \cdot \|x_m - T^n z_m\|
\right.
\]

\[
+ \frac{b_n}{1 - c_n} \cdot \|x_m - T^n z_m\| + \frac{1 + b_n + c_n}{1 - c_n} \cdot \sum_{i=0}^{n-1} \|T^{n+y} z_m - T^{n+y} z_m\| \right\}
\]

\[
\leq \frac{a_n + c_n}{1 - c_n} \cdot \|x_m - T^n z_m\|
\]

\[
+ \frac{b_n}{1 - c_n} \cdot \|x_m - T^n z_m\|
\]

\[
\leq \frac{a_n + c_n}{1 - b_n - c_n} \cdot r_m - 1.
\]

Taking the limit superior as \( i \to \infty \) on each side, by definition of \( z_m \), we get

\[
D_m \leq \lim_{i \to \infty} \left( \frac{a_n + c_n}{1 - b_n - c_n} \right) \cdot r_{m-1}
\]

\[
\leq \gamma \cdot r_{m-1}.
\]  

(4)

By (3) and (4), we obtain

\[
r_m \leq (\alpha + \beta) \cdot \gamma \cdot N(E) \cdot r_{m-1}
\]

\[
= A \cdot r_{m-1},
\]
where $A = (\alpha + \beta) \cdot \gamma \cdot N(E) < 1$ by the assumption of the theorem. Since
\[
\|z_{m+1} - z_m\| \leq r_m + D_m \to 0
\]
as $m \to \infty$, it follows that $z_m$ is a Cauchy sequence. Let $\lim_{m \to \infty} z_m = z \in K$. Then, we have
\[
\|z - T^m z\| \leq \|z - z_m\| + \|z_m - T^m z_m\| + \|T^m z_m - T^m z\|
\leq \|z - z_m\| + \|z_m - T^m z_m\| + \alpha_m \|z_m - z\|
+ b_m (\|z_m - T^m z_m\| + \|z - T^m z\|) + c_m (\|z_m - T^m z\| + \|z - T^m z_m\|)
\]
and so
\[
\|z - T^m z\| \leq \frac{1 + \alpha_m + 2c_m}{1 - b_m - c_m} \cdot \|z - z_m\| + \frac{1 + b_m + c_m}{1 - b_m - c_m} \cdot \|z_m - T^m z_m\|.
\]
Taking the limit superior as $i \to \infty$ on each side, we obtain
\[
\limsup_{i \to \infty} \|z - T^m z\| \leq \limsup_{i \to \infty} \frac{1 + \alpha_m + 2c_m}{1 - b_m - c_m} \cdot \|z - z_m\| + \limsup_{i \to \infty} \frac{1 + b_m + c_m}{1 - b_m - c_m} \cdot D_m \to 0
\]
as $m \to \infty$. Therefore we have $Tz = z$ by Lemma 1. This completes the proof.

If we put $b_m = c_m = 0$ in \(1\), then from Theorem 1, we have the following result.

COROLLARY 1. Let $K$ be a nonempty bounded closed convex subset of a Banach space $E$ which has uniformly normal structure, i.e. $N(E) < 1$. If $T : K \to K$ is an asymptotically regular mapping such that
\[
\liminf_{n \to \infty} \|T^n\| = k < [N(E)]^{-1},
\]
then $T$ has a fixed point in $K$.

REMARK. In place of bounded subset of $K$ in \(1\), we have weaker assumption that there is a $z_0$ in $K$ for which $\{T^n z_0\}$ is bounded.

REFERENCES

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