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ANGULAR ESTIMATIONS OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. The object of the present paper is to derive some argument properties of certain integral operators. Our results contain some interesting corollaries as the special cases.

KEY WORDS AND PHRASES: Argument, integral operators, starlike functions, Bazilević functions.
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1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. If f and g are analytic in U, we say that f is subordinate to g, written $f \prec g$, if there exists a Schwarz function $w(z)$ in U such that $f(z) = g(w(z))$. A function $f \in A$ is said to be in the class $S^*[E, F]$ if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Ez}{1+Fz} \, (z \in U, \, -1 \leq F < E \leq 1).$$

The class $S^*[E, F]$ was studied in [1,2]. In particular, $S^*[1-2\alpha, \, -1] \equiv S^*(\alpha)(0 \leq \alpha < 1)$ is the well known class of starlike functions of order α. We observe [2] that a function f is in $S^*[E, F]$ if and only if

$$\left| \frac{zf'(z)}{f(z)} - \frac{1-EF}{1-F^2} \right| < \frac{E-F}{1-F^2} \, (z \in U, F \neq \, -1) \tag{1.2}$$

and

$$Re\left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{1-E}{2} \, (z \in U, F = \, -1). \tag{1.3}$$

A function $f \in A$ is said to be in the class $B(\mu, \alpha, \beta)$ if it satisfies

$$Re\left\{ \frac{zf'(z)f^{\mu-1}}{g^{\mu}(z)} \right\} > \beta (z \in U)$$

for some $\mu(\mu > 0)$, $\beta(0 \leq \beta < 1)$ and $g \in S^*(\alpha)$. Furthermore, we denote $B_1(\mu, \alpha, \beta)$ by the subclass of $B(\mu, \alpha, \beta)$ for $g(z) \equiv z \in S^*(\alpha)$. The classes $B(\mu, \alpha, \beta)$ and $B_1(\mu, \alpha, \beta)$ are the subclasses of Bazilević functions in U [3]. We also note that $B(1, \alpha, \beta) \equiv C(\alpha, \beta)$ is an important subclass of close-to-convex functions [4].

For a positive real number $\mu > 0$ and a function $f \in A$, we define the integral operator $J_{c,\mu}$ by

$$J_{c,\mu}(f) = \left(\frac{c+\mu}{z^c}\int_0^z t^{c-1}f^{\mu}(t)dt\right)^{\frac{1}{\mu}} (c > -\mu). \tag{1 4}$$

Kumar and Shukla [5] showed that the integral operator $J_{c,\mu}(f)$ defined by (1.4) belongs to the class $S^*[E,F]$ for $c \geq \frac{\mu(E-1)}{1-F}$, whenever $f \in S^*[E,F]$. The operator $J_{c,1}$, when $c \in N = \{1,2,3,\cdots\}$, was introduced by Bernardi [6]. Further, the operator $J_{1,1}$ was studied earlier by Libera [7] and Livingston [8].

In the present paper, we give some argument properties of the integral operator defined by (1.4). We also generalize the previous results of Libera [7], Owa and Srivastava [9] and Owa and Obradović [10].

2. MAIN RESULTS

In proving our main results, we shall need the following lemmas.

LEMMA 1 ([11]). Let $M(z)$ and $N(z)$ be regular in U with $M(0) = N(0) = 0$, and let β be real. If $N(z)$ maps U onto a (possibly many-sheeted) region which is starlike with respect to the origin, then

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} > \beta(z \in U) \Rightarrow Re\left\{\frac{M(z)}{N(z)}\right\} > \beta(z \in U)$$

and

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} < \beta(z \in U) \Rightarrow Re\left\{\frac{M(z)}{N(z)}\right\} < \beta(z \in U).$$

LEMMA 2 ([12]). Let $p(z)$ be analytic in U, $p(0) = 1$, $p(z) \neq 0$ in U and suppose that there exists a point $z_0 \in U$ such that

$$\left|arg\, p(z)\right| < \frac{\pi\beta}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$\left|arg\, p(z_0)\right| = \frac{\pi\beta}{2},$$

where $\beta > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad arg\, p(z_0) = \frac{\pi\beta}{2}$$

and

$$k \leq -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad arg\, p(z_0) = -\frac{\pi\beta}{2}$$

where

$$p(z_0)^{\frac{1}{\beta}} = \pm ia (a > 0).$$

With the help of Lemma 1 and Lemma 2, we now derive

THEOREM 1. Let c and μ be real numbers with $c \geq 0$, $\mu > 0$ and $-1 \leq F < E \leq 1$ and let $f \in A$ If

$$\left|arg\left(\frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)}-\beta\right)\right|<\frac{\pi\delta}{2}\ (0\leq\beta<1, 0<\delta\leq 1)$$

for some $g\in S^*[E,F]$, then

$$\left|arg\left(\frac{z(J_{c,\mu}(f))'J_{c,\mu}^{\mu-1}(f)}{J_{c,\mu}^{\mu}(g)}\right)-\beta\right|<\frac{\pi\eta}{2},$$

where $J_{c,\mu}$ is the integral operator defined by (1.4) and $\eta(0<\eta\leq 1)$ is the solution of the equation

$$\delta=\begin{cases}\eta+\dfrac{2}{\pi}Tan^{-1}\left(\dfrac{\eta\, sin\,\frac{\pi}{2}(1-t_c(E,F))}{c+\frac{1+E}{1+F}+\eta\, cos\,\frac{\pi}{2}(1-t_c(E,F))}\right) & \text{for } F\neq -1,\\ \eta & \text{for } F=-1,\end{cases}\tag{2.1}$$

when

$$t_c(E,F)=\frac{2}{\pi}sin^{-1}\left(\frac{E-F}{c(1-F^2)+1-EF}\right).\tag{2.2}$$

PROOF. Let us put

$$p(z)=\frac{M(z)}{N(z)},$$

where

$$M(z)=\frac{1}{1-\beta}\left\{z^c f^{\mu}(z)-c\int_0^z t^{c-1}f^{\mu}(t)dt-\beta\mu\int_0^z t^{c-1}g^{\mu}(t)dt\right\}$$

and

$$N(z)=\mu\int_0^z t^{c-1}g^{\mu}(t)dt.$$

Then $p(z)$ is analytic in U with $p(0)=1$. By a simple calculation, we have

$$\begin{aligned}\frac{M'(z)}{N'(z)}&=p(z)\left(1+\frac{N(z)}{zN'(z)}\frac{zp'(z)}{p(z)}\right)\\&=\frac{1}{1-\beta}\left(\frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)}-\beta\right).\end{aligned}$$

Since $g\in S^*[E,F]$, $J_{c,\mu}(g)\in S^*[E,F]$ [5] and hence $N(z)$ is (possibly many-sheeted) starlike function with respect to the origin. Therefore, from our assumption and Lemma 1, $p(z)\neq 0$ in U.

If there exists a point $z_0\in U$ such that

$$|arg\, p(z)|<\frac{\pi\eta}{2}\quad\text{for}\quad |z|<|z_0|$$

and

$$|arg\, p(z_0)|=\frac{\pi\eta}{2},$$

then, from Lemma 2, we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\eta,$$

where

$$k\geq\frac{1}{2}\left(a+\frac{1}{a}\right)\quad\text{when}\quad arg\, p(z_0)=\frac{\pi\eta}{2}$$

and

$$k \leq -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad arg\, p(z_0) = -\frac{\pi\eta}{2}$$

where

$$p(z_0)^{\frac{1}{\eta}} = \pm ia (a > 0).$$

Since $J_{c,\mu}(g) \in S^*[E,F]$, from (1.2) and (1.3), we have

$$\frac{zN'(z)}{N(z)} = \frac{z(J_{c,\mu}(g))'}{J_{c,\mu}(g)} + c = \rho e^{i\frac{\pi\theta}{2}},$$

where

$$\begin{cases} c + \dfrac{1-E}{1-F} < \rho < c + \dfrac{1+E}{1+F}, \\ -t_c(E,F) < \phi < t_c(E,F) \quad \text{for } F \neq -1, \end{cases}$$

when $t_c(E,F)$ is given by (2.2), and

$$\begin{cases} c + \dfrac{1-E}{2} < \rho < \infty, \\ -1 < \phi < 1 \quad \text{for } F = -1. \end{cases}$$

At first, suppose that $p(z_0)^{\frac{1}{\eta}} = ia (a > 0)$. For the case $F \neq -1$, we obtain

$$\begin{aligned} arg\left(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\right) &= arg\, \frac{(1-\beta)M'(z_0)}{N'(z_0)} \\ &= arg\, p(z_0) + arg\left(1 + \frac{1}{\frac{z(J_{c,\mu}(g))'}{J_{c,\mu}(g)} + c}\, \frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= \frac{\pi\eta}{2} + arg\left(1 + \left(\rho e^{i\frac{\pi\phi}{2}}\right)^{-1} i\eta k\right) \\ &= \frac{\pi\eta}{2} + Tan^{-1}\left(\frac{\eta k\, sin\, \frac{\pi}{2}(1-\phi)}{\rho + \eta k\, cos\, \frac{\pi}{2}(1-\phi)}\right) \\ &\geq \frac{\pi\eta}{2} + Tan^{-1}\left(\frac{\eta\, sin\, \frac{\pi}{2}(1-t_c(E,F))}{c + \frac{1+E}{1+F} + \eta\, cos\, \frac{\pi}{2}(1-t_c(E,F))}\right) \\ &= \frac{\pi}{2}\delta, \end{aligned}$$

where $t_c(E,F)$ and δ are given by (2.2) and (2.1), respectively. Similarly, for the case $F = -1$, we have

$$arg\left(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\right) \geq \frac{\pi\eta}{2}.$$

These are a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\eta}} = -ia (a > 0)$. For the case $F \neq -1$, applying the same method as the above, we have

$$arg\left(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\right) = \leq -\frac{\pi\eta}{2} - Tan^{-1}\left(\frac{n\, sin\, \frac{\pi}{2}(1-t_c(E,F))}{c + \frac{1+E}{1+F} + \eta\, cos\, \frac{\pi}{2}(1-t_c(E,F))}\right)$$

where $t_c(E,F)$ and δ are given by (2.2) and (2.1), respectively and for the case $F = -1$, we have

$$arg\left(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\right) \leq -\frac{\pi\eta}{2},$$

which are contradictions to the assumption. Therefore we complete the proof of our theorem.

Taking $E = 1 - 2\alpha (0 \leq \alpha < 1)$ and $F = -1$ in Theorem 1, we have

COROLLARY 1. Let $c \geq 0$, $\mu > 0$ and $f \in A$. If

$$\left| arg\left(\frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)} - \beta\right)\right| < \frac{\pi\delta}{2} \, (0 \leq \beta < 1, 0 < \delta \leq 1)$$

for some $g \in S^*(\alpha)$, then

$$\left| arg\left(\frac{z(J_{c,\mu}(f))' J_{c,\mu}^{\mu-1}(f)}{J_{c,\mu}^{\mu}(g)} - \beta\right)\right| < \frac{\pi\delta}{2},$$

where $J_{c,\mu}$ is the integral operator defined by (1.4).

REMARK 1. For $\delta = 1$, Corollary 1 is the result obtained by Owa and Obradović [10].

Setting $E = 1$, $F = -1$, $\mu = 1$, $\delta = 1$ and $g(z) = z$ in Theorem 1, we have

COROLLARY 2. Let $c \geq 0$ and $f \in A$. If

$$Re\, f'(z) > \beta (0 \leq \beta < 1),$$

then

$$Re\,(J_{c,1}(f))' > \beta,$$

where $J_{c,1}$ is the integral operator defined by (1.4).

Letting $\mu = 1$ in Theorem 1, we have

COROLLARY 3. Let $c \geq 0$ and $-1 \leq F < E \leq 1$ and let $f \in A$. If

$$\left| arg\left(\frac{zf'(z)}{g(z)} - \beta\right)\right| < \frac{\pi\delta}{2} \, (0 \leq \beta < 1, 0 < \delta \leq 1)$$

for some $g \in S^*[E, F]$, then

$$\left| arg\left(\frac{z(J_{c,1}(f))'}{J_{c,1}(g)} - \beta\right)\right| < \frac{\pi\eta}{2},$$

where $J_{c,1}$ is the integral operator defined by (1.4) and $\eta (0 < \eta \leq 1)$ is the solution of the equation (2.1).

Taking $E = 1 - 2\alpha (0 \leq \alpha < 1)$ and $F = -1$ in Corollary 3, we have

COROLLARY 4. Let $c \geq 0$ and $f \in A$. If

$$\left| arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi\delta}{2} \, (0 \leq \alpha < 1, 0 < \delta \leq 1),$$

then

$$\left| arg\left(\frac{z(J_{c,1}(f))'}{J_{c,1}(f)} - \alpha\right)\right| < \frac{\pi\delta}{2},$$

where $J_{c,1}$ is the integral operator defined by (1.4).

Putting $E = 1 - 2\alpha (0 \leq \alpha < 1)$, $F = -1$ and $\delta = 1$ in Corollary 3 and Corollary 4, we obtain the following result of Owa and Srivastava [9].

COROLLARY 5. If the function f defined by (1.1) is in the class $C(\alpha, \beta)$, then the integral operator $J_{c,1}(f)(c \geq 0)$ defined by (1.4) is also in the class $c(\alpha, \beta)$.

REMARK 2. Taking $\alpha = \beta = 0$ and $c = 1$ in Corollary 5, we obtain the result given earlier by Libera [7]

By using the same technique as in proving Theorem 1, we have

THEOREM 2. Let c and μ be real numbers with $c \geq 0$, $\mu > 0$ and $-1 \leq F < E \leq 1$ and let $f \in A$. If

$$\left|arg\left(\beta - \frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)}\right)\right| < \frac{\pi\delta}{2}\,(\beta > 1, 0 < \delta \leq 1)$$

for some $g \in S^*[E,F]$, then

$$\left|arg\left(\beta - \frac{z(J_{c,\mu}(f))'J_{c,\mu}^{\mu-1}(f)}{J_{c,\mu}^{\mu}(g)}\right)\right| < \frac{\pi\eta}{2},$$

where $J_{c,\mu}$ is the integral operator defined by (1.4) and $\eta(0 < \eta \leq 1)$ is the solution of the equation (2.1)

Putting $E = 1 - 2\alpha(0 \leq \alpha < 1)$, $F = -1$, $\mu = 1$ and $\delta = 1$ in Theorem 2, we have the following result by Owa and Srivastava [9].

COROLLARY 6. Let $c \geq 0$ and $f \in A$. If

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} < \beta(\beta > 1)$$

for some $g \in S^*(\alpha)$, then

$$Re\left\{\frac{z(J_{c,1}(f))'}{J_{c,1}(g)}\right\} < \beta,$$

where $J_{c,1}$ is the integral operator defined by (1.4).

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REFERENCES

[1] JANOWSKI, W., Some extremal problems for certain families of analytic functions, *Bull. de L'Acad. Pol. des Sci.* **21** (1973), 17-25.

[2] SILVERMAN, H. and SILVIA, E.M., Subclasses of starlike functions subordinate to convex functions, *Can. J. Math.* **37** (1985), 48-61

[3] SINGH, R., On Bazilević functions, *Proc. Amer. Math. Soc.* **1** (1952), 169-815.

[4] KAPLAN, W., Close-to-convex Schlicht functions, *Michigan Math. J.* **1** (1952), 169-185.

[5] KUMAR, V. and SHUKLA, S.L., On p-valent starlike functions with reference to the Bernardi integral operator, *Bull. Austral. Math. Soc.* **30** (1984), 37-43.

[6] BERNARDI, S.D., Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* **135** (1969), 429-446.

[7] LIBERA, R.J., Some classes of regular univalent functions, *Proc. Amer. Math. Soc.* **16** (1965), 755-758

[8] LIVINGSTIN, A.E., On the radius of univalence of certain analytic functions, *Proc. Amer. Math. Soc.* **17** (1966), 352-357.

[9] OWA, S. and SRIVASTAVA, H.M., Some applications of the generalized Libera integral operator, *Proc. Japan Acad.* **62**, Ser. A (1986), 125-128.

[10] OWA, S. and OBRADOVIĆ, M., Certain subclasses of Bazilević functions of type α, *Internat. J. Math. Math. Sci.* **9** (1986), 347-359.

[11] MILLER, S.S. and MOCANU, P.T., Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.* **65** (1978), 289-305.

[12] NUNOKAWA, M., On the order of strongly starlikeness of strongly convex functions, *Proc. Japan Acad.* **69**, Ser. A (1993), 234-237.