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A NOTE ON COMMUTATIVITY OF AUTOMORPHISMS

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ABSTRACT. Let α and β be automorphisms of a ring satisfying the equation $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ In this paper we prove some results where this equation itself implies the commutativity of α and β.

KEY WORDS AND PHRASES: Unital ring, nilpotent element, automorphism
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1. INTRODUCTION

The equation

$$\alpha+\alpha^{-1}=\beta+\beta^{-1}, \tag{$*$}$$

where α and β are automorphisms of a ring R, has been of considerable interest during recent years (see e.g [1,2]). The study of this equation becomes simpler when an additional assumption of commutativity of α and β is made. However, some situations have been identified where equation $(*)$ itself implies the commutativity of α and β. For instance, it has been shown in [1, Corollary 3] that if R is a semiprime unital ring containing the element $1/2$ and α,β are inner automorphisms os R satisfying the equation $(*)$, then α and β commute.

The purpose of this note is precisely to address the commutativity problém and prove certain results in this context. The main result (Theorem 2.1) is, in fact, a generalization of a result of Cater and Thaheem [3] proved for complex algebras, where the equation $(*)$ appears in a more general form $\alpha+m\alpha^{-1}=\beta+m\beta^{-1}$ for an appropriate integer m The mappings of the type $\alpha+m\alpha^{-1}$ occur in the studies of automorphisms of certain C^*-algebras (see for instance [3]). We show here (Theorem 2 1) that if R is a unital ring and α and β are inner automorphisms of R induced by u and v, respectively, such that (i) $\alpha(x)+m\alpha^{-1}(x)=\beta(x)+m\beta^{-1}(x)$, for $x=u,v$, and (ii) $\alpha\beta(x)=\beta\alpha(x)$, for $x=u,v$, where R is (m^2-1)-torsion free, then α and β commute As a corollary (Corollary 2 2) we provide an alternate proof of a special case of Brešar's result [1, Corollary 3] when R has no nontrivial nilpotent elements. However as in Theorem 2.1, the equation $(*)$ here need not hold for all the elements of the ring

We remark that the equation $(*)$ has been extensively studied for von Neumann algebras and C^*-algebras and for more information in this context we may refer to [4,5], which contain further references.

2. COMMUTATIVITY RESULTS

We begin with the following theorem which generalizes a result of Cater and Thaheem [3] proved for complex algebras Our approach here is almost analogous to that of [3]

THEOREM 2.1. Let R be a unital ring and α,β be inner automorphisms of R induced by u and v, respectively, such that

$$\alpha(x) + m\alpha^{-1}(x) = \beta(x) + m\beta^{-1}(x), \quad \text{for} \quad x = u, v, \tag{i}$$

and

$$\alpha\beta(x) = \beta\alpha(x), \quad \text{for} \quad x = u, v. \tag{ii}$$

If R is $(m^2 - 1)$-torsion free, then α and β commute.

PROOF. Put $k = u^{-1}vuv^{-1}$ We first show that k commutes with u and v Substituting $x = v$ in (ii), we get $uvu^{-1} = vuvu^{-1}v^{-1}$ We may rewrite this equation as

$$vu = uvk \tag{1}$$

or

$$vk = u^{-1}vu. \tag{2}$$

Also,

$$kv = u^{-1}vuv^{-1}v = u^{-1}vu. \tag{3}$$

It follows from (2) and (3) that $kv = vk$ Thus k and v commute By symmetry, k and u also commute Thus we can write (1) as

$$kuv = vu. \tag{4}$$

Substituting $x = v$ in (i), we get

$$uvu^{-1} + mu^{-1}vu = (1 + m)v. \tag{5}$$

Multiplying (5) on the right by u, we get

$$uv + mu^{-1}vuu = (1 + m)vu. \tag{6}$$

It follows from (4) and (6) that

$$uv + mkvu = (1 + m)vu. \tag{7}$$

It follows from (4) and (7) that

$$uv + mk^2uv = (1 + m)kuv, \tag{8}$$

or what is the same

$$(mk - 1)(k - 1)uv = \big(mk^2 - mk - k + 1\big)uv = 0. \tag{9}$$

Since uv is invertible, we get from (9) that

$$mk^2 - mk - k + 1 = 0. \tag{10}$$

We observe from (4) that $k^{-1}vu = uv$. Repeating the above procedure with v in place of u, u in place of v and k^{-1} in place of k, we get

$$\big(mk^{-1} - 1\big)\big(k^{-1} - 1\big) = 0. \tag{11}$$

Multiplying (11) on the left by mk and on the right by k, we obtain

$$\big(m^2 - mk\big)(1 - k) = m^2 - m^2k - mk + mk^2 = 0, \tag{12}$$

or what is the same

$$- mk^2 + mk + m^2k - m^2 = 0. \tag{13}$$

Adding (10) and (13), we get

$$(m^2 - 1)(k - 1) = 0. \tag{14}$$

Since R is $(m^2 - 1)$-torsion free, we get from (14) that $k - 1 = 0$ or $k = 1$ This implies $uv = vu$ and hence α and β commute

The following corollary gives an alternate proof of Brešar's result [1, Corollary 3] in the special case when R has no nontrivial nilpotent elements with an additional assumption that $(\alpha\beta)(u) = (\beta\alpha)(u)$ and $(\alpha\beta)(v) = (\alpha\beta)(v)$ However, in our setting it is sufficient for equation $(*)$ to hold for some specific elements rather than all the elements of the ring to ensure the commutativity of α and β

COROLLARY 2.2. Let R be a unital ring with no nonzero nilpotent elements and α,β be inner automorphisms of R induced by u and v respectively such that

$$\alpha(v) + \alpha^{-1}(v) = \beta(v) + \beta^{-1}(v) \tag{iii}$$

and

$$\alpha\beta(x) = \beta\alpha(x), \quad \text{for} \quad x = u, v. \tag{iv}$$

Then α and β commute

PROOF. As in the proof of Theorem 2 1, put $k = u^{-1}vuv^{-1}$ Then k commutes with u and v follows from (iv) and consequently equation (4) holds Then using (iii) and following a procedure similar to Theorem 2 1, we obtain that $(k - 1)^2 = 0$ Since R has no nonzero nilpotent elements, we get $k - 1 = 0$ or $k = 1$. This proves that $uv = vu$ and hence α and β commute

REMARK 2.3. (a) We observe that the main argument in proving the commutativity of α and β has been to show that $k = 1$. In fact, somewhat weaker condition, namely that k belongs to the center of R would also ensure the commutativity of α and β Therefore, it would also be interesting to prove that k is in the center of R

(b) It would be interesting to prove or disprove Theorem 2.1 for the case $m = -1$.

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