A REMARK ON 9-REGULAR SPACES

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ABSTRACT. In this paper we give an embedding characterization of θ -regularity using the Wallman-type compactification. The productivity of θ -regularity and a slight generalization of Nagami's Product Theorem to non-Hausdorff paracompact Σ -spaces we obtain as a corollary.

KEY WORDS AND PHRASES. θ -regularity. Wallman-type compactification, paracompact Σ -space

AMS SUBJECT CLASSIFICATION CODES: Primary 54A20, 54D35. Secondary 54B10, 54D20.

1. PRELIMINARIES

A filter base Φ in a topological space X has a θ -cluster point $x \in X$ if every closed neighborhood H of x and every $F \in \Phi$ have a nonempty intersection. The filter base Φ θ -converges to its θ -limit x if for every closed neighborhood H of x there is $F \in \Phi$ such that $F \subseteq H$. Recall that a topological space X is said to be θ -regular [3] if every filter base in X with a θ -cluster point has a cluster point. A topological space is said to be a Σ -space [1] if there exist locally finite closed collections Φ_i , $i=1,2,\ldots$ and a cover Γ which consists of closed countably compact sets such that if $C \in \Gamma$ and $C \subseteq U$, where U is open in X, then $C \subseteq F \subseteq U$ for some $i \in \mathbb{N}$, $F \in \Phi_i$. A topological space is called (semi-) paracompact, if every its open cover has an open $(\sigma$ -) locally finite refinement. Paracompact spaces are θ -regular [4].

Let X be a topological space with $\mathfrak G$ its closed base which is a *lattice* (that means $\varnothing, X \in \mathfrak G$ and $\mathfrak G$ contains all its finite unions and intersections). Recall that the *Wallman-type* [2] or \check{S} ann [6] compactification is defined as the set $\omega(X,\mathfrak G)=X\cup\{y|y\text{ is an ultra-}\mathfrak G$ filter in X with no cluster point}, where the term "ultra- $\mathfrak G$ " means maximal among all filters with a base consisting of elements from $\mathfrak G$. The set $\omega(X,\mathfrak G)$ can be topologized by the open base consisting of the sets $S(U)=U\cup\{y|y\in\omega(X,\mathfrak G)\setminus X,U\in y\}$ where $X\setminus U\in \mathfrak G$. If $\mathfrak C$ is the collection of all closed sets in X then $\omega(X,\mathfrak C)=\omega X$ is the Wallman compactification of X.

2. MAIN RESULTS

Let X be a topological space with a closed base \mathfrak{G} . We say that \mathfrak{G} is balanced if \mathfrak{G} is a lattice and every $x \in X$ has a neighborhood base, say δ_x , such that $\operatorname{cl} U \in \mathfrak{G}$ for every $U \in \delta_x$. Trivially, the collection \mathfrak{C} of all closed sets of X is balanced. Two disjoint sets $A, B \subseteq X$ are said to be

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point-wise separated in X if every $x \in A$, $y \in B$ have open disjoint neighborhoods. Now, we can state the theorem.

Theorem 1. Let X be a topological space with a balanced closed base \mathfrak{G} . The following statements are equivalent.

- (i) X is θ -regular
- (ii) The sets X, $\omega(X, \mathfrak{G}) \setminus X$ are point-wise separated in $\omega(X, \mathfrak{G})$.
- (iii) There exists a compact space K containing X as a subspace such that the sets X, $K \setminus X$ are point-wise separated in K.

Proof. Suppose (i). Let $x \in X$ and $y \in \omega(X, \mathfrak{G}) \setminus X$. Since X is θ -regular the filter y has no θ -cluster point. It follows that x has an open neighborhood U with $\operatorname{cl} U \in \mathfrak{G}$ such that $F \cap \operatorname{cl} U = \emptyset$ for some $F \in y$. Then $V = X \setminus \operatorname{cl} U \in y$ and, consequently, $y \in S(V)$. Now, let $W \subseteq U$ be an open neighborhood of x with $X \setminus W \in \mathfrak{G}$. One can easily check that S(W), S(V) are disjoint neighborhoods of the points x, y. It follows (ii).

(ii) \Longrightarrow (iii) is trivial Suppose (iii). Let Φ be a filter base with a θ -cluster point $x \in X$. There exists a filter base Φ' finer than Φ which θ -converges to x. Since K is compact, Φ' has some cluster point $y \in K$. But a θ -limit and a cluster point of the same filter base cannot have disjoint neighborhoods; hence $y \in X$. Finally, y is a cluster point of Φ which implies (i).

Corollary 1. The product of θ -regular topological spaces is θ -regular.

Proof. Let X_{α} , $\alpha \in A$ be θ -regular topological spaces. It follows from the Theorem 1 that there are compact spaces $K_{\alpha} \supseteq X_{\alpha}$ such that for every $\alpha \in A$ the sets X_{α} , $K_{\alpha} \setminus X_{\alpha}$ are point-wise separated. Let $K = \prod_{\alpha \in A} K_{\alpha}$, $X = \prod_{\alpha \in A} X_{\alpha}$. Then K is compact and, evidently, the sets X, $K \setminus X$ are point-wise separated. Hence, the space X is θ -regular.

K. Nagami in [5] proved that a countable product of paracompact Hausdorff Σ -spaces is paracompact. Nagami uses Hausdorff separation axiom for upgrading semiparacompactness to paracompactness. However, Nagami's proof essentially contains the result that a countable product of paracompact Σ -spaces is semiparacompact which needs no separation axioms. The following result now follows from the fact that θ -regular semiparacompact spaces are paracompact ([4], Theorem 6).

Corollary 2. A countable product of paracompact (not necessarily regular or Hausdorff) Σ-spaces is paracompact.

It is easy to check that a second countable space has a countable balanced closed base. Theorem 1 (with Theorem 6, [4]) also yields the following.

Corollary 3. A topological space X is paracompact second countable if and only if there exists a compact second countable space K containing X as a subspace such that the sets X, $K \setminus X$ are point-wise separated in K.

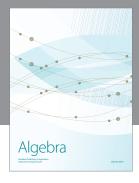
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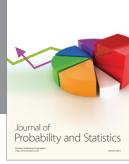
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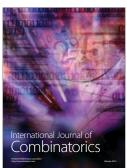








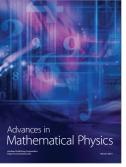






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