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A REMARK ON Θ-REGULAR SPACES

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ABSTRACT. In this paper we give an embedding characterization of θ-regularity using the Wallman-type compactification. The productivity of θ-regularity and a slight generalization of Nagami's Product Theorem to non-Hausdorff paracompact Σ-spaces we obtain as a corollary.

KEY WORDS AND PHRASES. θ-regularity, Wallman-type compactification, paracompact Σ-space

AMS SUBJECT CLASSIFICATION CODES: Primary 54A20, 54D35. Secondary 54B10, 54D20.

1. PRELIMINARIES

A filter base Φ in a topological space X has a *θ-cluster point* $x \in X$ if every closed neighborhood H of x and every $F \in \Phi$ have a nonempty intersection. The filter base Φ *θ-converges* to its *θ-limit* x if for every closed neighborhood H of x there is $F \in \Phi$ such that $F \subseteq H$. Recall that a topological space X is said to be *θ-regular* [3] if every filter base in X with a θ-cluster point has a cluster point. A topological space is said to be a *Σ-space* [1] if there exist locally finite closed collections Φ_i, $i = 1, 2, \ldots$ and a cover Γ which consists of closed countably compact sets such that if $C \in \Gamma$ and $C \subseteq U$, where U is open in X, then $C \subseteq F \subseteq U$ for some $i \in \mathbb{N}$, $F \in \Phi_i$. A topological space is called *(semi-) paracompact*, if every its open cover has an open (σ-) locally finite refinement. Paracompact spaces are θ-regular [4].

Let X be a topological space with $\mathfrak{G}$ its closed base which is a *lattice* (that means $\varnothing, X \in \mathfrak{G}$ and $\mathfrak{G}$ contains all its finite unions and intersections). Recall that the *Wallman-type* [2] or *Šanin* [6] compactification is defined as the set $\omega(X, \mathfrak{G}) = X \cup \{y \mid y$ is an ultra-$\mathfrak{G}$ filter in X with no cluster point$\}$, where the term "ultra-$\mathfrak{G}$" means maximal among all filters with a base consisting of elements from $\mathfrak{G}$. The set $\omega(X, \mathfrak{G})$ can be topologized by the open base consisting of the sets $S(U) = U \cup \{y \mid y \in \omega(X, \mathfrak{G}) \setminus X, U \in y\}$ where $X \setminus U \in \mathfrak{G}$. If $\mathfrak{C}$ is the collection of all closed sets in X then $\omega(X, \mathfrak{C}) = \omega X$ is the Wallman compactification of X.

2. MAIN RESULTS

Let X be a topological space with a closed base $\mathfrak{G}$. We say that $\mathfrak{G}$ is *balanced* if $\mathfrak{G}$ is a lattice and every $x \in X$ has a neighborhood base, say δ_x, such that $\mathrm{cl}\, U \in \mathfrak{G}$ for every $U \in \delta_x$. Trivially, the collection $\mathfrak{C}$ of all closed sets of X is balanced. Two disjoint sets $A, B \subseteq X$ are said to be *point-wise separated in* X if every $x \in A$, $y \in B$ have open disjoint neighborhoods. Now, we can state the theorem.

Theorem 1. *Let* X *be a topological space with a balanced closed base* $\mathfrak{G}$. *The following statements are equivalent.*

(i) X *is* θ*-regular*

(ii) *The sets* X, $\omega(X, \mathfrak{G}) \setminus X$ *are point-wise separated in* $\omega(X, \mathfrak{G})$.

(iii) *There exists a compact space* K *containing* X *as a subspace such that the sets* X, $K \setminus X$ *are point-wise separated in* K.

Proof. Suppose (i). Let $x \in X$ and $y \in \omega(X, \mathfrak{G}) \setminus X$. Since X is θ-regular the filter y has no θ-cluster point. It follows that x has an open neighborhood U with $\operatorname{cl} U \in \mathfrak{G}$ such that $F \cap \operatorname{cl} U = \varnothing$ for some $F \in y$. Then $V = X \setminus \operatorname{cl} U \in y$ and, consequently, $y \in S(V)$. Now, let $W \subseteq U$ be an open neighborhood of x with $X \setminus W \in \mathfrak{G}$. One can easily check that $S(W)$, $S(V)$ are disjoint neighborhoods of the points x, y. It follows (ii).

(ii) $\Longrightarrow$ (iii) is trivial. Suppose (iii). Let Φ be a filter base with a θ-cluster point $x \in X$. There exists a filter base Φ' finer than Φ which θ-converges to x. Since K is compact, Φ' has some cluster point $y \in K$. But a θ-limit and a cluster point of the same filter base cannot have disjoint neighborhoods; hence $y \in X$. Finally, y is a cluster point of Φ which implies (i).

Corollary 1. *The product of* θ*-regular topological spaces is* θ*-regular.*

Proof. Let X_α, $\alpha \in A$ be θ-regular topological spaces. It follows from the Theorem 1 that there are compact spaces $K_\alpha \supseteq X_\alpha$ such that for every $\alpha \in A$ the sets X_α, $K_\alpha \setminus X_\alpha$ are point-wise separated. Let $K = \prod_{\alpha \in A} K_\alpha$, $X = \prod_{\alpha \in A} X_\alpha$. Then K is compact and, evidently, the sets X, $K \setminus X$ are point-wise separated. Hence, the space X is θ-regular.

K. Nagami in [5] proved that a countable product of paracompact Hausdorff Σ-spaces is paracompact. Nagami uses Hausdorff separation axiom for upgrading semiparacompactness to paracompactness. However, Nagami's proof essentially contains the result that a countable product of paracompact Σ-spaces is semiparacompact which needs no separation axioms. The following result now follows from the fact that θ-regular semiparacompact spaces are paracompact ([4], Theorem 6).

Corollary 2. *A countable product of paracompact (not necessarily regular or Hausdorff)* Σ*-spaces is paracompact.*

It is easy to check that a second countable space has a countable balanced closed base. Theorem 1 (with Theorem 6 , [4]) also yields the following.

Corollary 3. *A topological space* X *is paracompact second countable if and only if there exists a compact second countable space* K *containing* X *as a subspace such that the sets* X, $K \setminus X$ *are point-wise separated in* K.

References

1. Burke D. K., *Covering properties, Handbook of Set-theoretic Topology*, North-Holland, Amsterdam, 1984.
2. Császár A., *General Topology*, Akademiai Kiadó, Budapest, 1978.
3. Janković D. S., *θ-regular spaces*, preprint, *published:* Internat. J. Math. Sci. **8** (1985), no.3, 615-619.
4. Kovár M. M., *On θ-regular spaces*, Internat. J. Math. Sci. **17** (1994), no.4, 687-692.
5. Nagami K., *Σ-spaces*, Fund. Math. **65** (1969), 169-192.
6. Nagata J., *Modern General Topology*, North-Holland, Amsterdam, 1974.