ABSTRACT. In this paper, We prove that every \((\epsilon)\)-sasakian manifold is a hypersurface of an indefinite kaehlerian manifold, and give a necessary and sufficient condition for a Riemannian manifold to be an \((\epsilon)\)-sasakian manifold.

KEY WORDS AND PHRASES: \((\epsilon)\)-sasakian manifolds; real hypersurface; indefinite kaehlerian manifolds; \((\epsilon)\)-almost contact structure.

1. INTRODUCTION Let \(M\) be a real \((2n + 1)\)-dimensional differentiable manifold endowed with an almost contact structure \((\phi, \xi, \eta)\). This means that \(\phi\) is a tensor field of type \((1,1)\), \(\xi\) is a vector field and \(\eta\) is a 1-form on \(M\) satisfying:

\[
\phi^2 = -I + \eta \otimes \xi; \quad \eta(\xi) = 1
\]

It follows that

\[
\eta \circ \phi = 0; \phi(\xi) = 0; \text{rank}\phi = 2n
\]

If there exists a semi-Riemannian metric \(g\) on \(M\) that satisfies (see [1])

\[
g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y) \quad \forall X, Y \in \Gamma(TM)
\]

Where \(\epsilon = \pm 1\), We call \((\phi, \xi, \eta, g)\) an \((\epsilon)\)-almost contact metric structure and \(M\) an \((\epsilon)\)-almost contact metric manifold.

From (3), we have

\[
\eta(X) = \epsilon g(X, \xi) \quad \forall X \in \Gamma(TM)
\]

\[
g(\xi, \xi) = \epsilon
\]

We say that \((\phi, \xi, \eta, g)\) is an \((\epsilon)\)-contact metric structure if we have

\[
g(X, \phi Y) = d\eta(X, Y) \quad \forall X, Y \in \Gamma(TM)
\]
In this case, $M$ is an $(e)$-contact metric manifold. An $(e)$-contact metric structure which is normal is called an $(e)$-sasakian structure. A manifold endowed with an $(e)$-sasakian structure is called an $(e)$-sasakian manifold.

In [1], A. Bejancu and K.L. Duggal give a theorem as following:

**THEOREM A** (see [1] theorem 6)

Let $M$ be an orientable real hypersurface of an indefinite kaehlerian manifold $\overline{M}$, then the following assertions with respect to the $(e)$-almost contact metric structure inherited by $M$ are equivalent:

1. $M$ is an $(e)$-sasakian manifold
2. The $(e)$-characteristic vector field satisfies
   \[ Vx - XfVx \subset (TM) \]
3. The shape operator $A$ satisfies
   \[ AX = -eX + (e + \eta(Ax))\eta(X)\xi \quad \forall X \in \Gamma(TM) \]

This produces a problem whether an $(e)$-sasakian manifold must be a real hypersurface of some indefinite kaehlerian manifold. In sec.2, we prove that the answer to this problem is positive. That is

**THEOREM 1.1.** Every $(e)$-sasakian manifold must be a real hypersurface of some indefinite kaehlerian manifold.

In [2], Hatakeyama, Ogewa and Tanno give the condition for a Riemannian manifold to be a $K$-contact manifold, they prove

**THEOREM B** (see [2] or [4]) In order that a $(2n + 1)$-dimensional Riemannian manifold $M$ is $K$-contact, it is necessary and sufficient that the following two conditions are satisfied:

1. $M$ admits a unit killing vector field $\xi$;
2. The sectional curvatures for plane sections containing $\xi$ are equal to 1 at every point of $M$.

In sec.3, we generalize Theorem B by giving the necessary and sufficient condition for a Riemannian manifold to be an $(e)$-sasakian manifold, that is

**THEOREM 1.2.** In order that a $(2n + 1)$-dimensional Riemannian manifold $M$ is $(e)$-sasakian manifold, it is necessary and sufficient that the following three conditions are satisfied:

1. $M$ admits a unit killing vector field $\xi$;
2. The sectional curvature for plane sections containing $\xi$ are equal to 1 or -1 at every point on $M$.
3. $R(X, Y)\xi = 0 \quad \forall X, Y \perp \xi$

2. THE PROOF OF THEOREM 1.1

Let $M$ be a $(2n + 1)$-dimensional $(e)$-sasakian manifold with $(e)$-sasakian structure $(\phi, \xi, \eta, g)$. Let $R$ be real line with coordinate $t$ and unit tangent vector $\frac{d}{dt}$. Denote $M \times R$ by $\overline{M}$, then vector fields on $\overline{M}$ are given by $\overline{X} = (X, f\frac{d}{dt}), \overline{Y} = (Y, h\frac{d}{dt}), \ldots$, 
Where $X, Y \ldots$ are vector fields tangent to $M$ and $f, h, \ldots$ are function on $M$, we define a linear map $J$ on the tangent space of $\overline{M}$ by [5]

$$J\overline{\overline{X}} = J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X) \frac{d}{dt})$$

(7)

From (1) and (2), we have

$$J^2 \overline{\overline{X}} = J(\phi X - f \xi, \eta(X) \frac{d}{dt}) = (\phi^2 X - \eta(X) \xi, -f \frac{d}{dt}) = -\overline{\overline{X}}$$

It shows that $J$ is almost complex structure on $\overline{M}$, but $M$ is an $(\varepsilon)$-sasakian manifold, this means $N(J) = 0$, then $J$ is a complex structure on $\overline{M}$, thus $\overline{\overline{M}} = M \times R$ is a complex manifold.

Let $\pi : \overline{\overline{M}} = M \times R \to M$ be the projection map, we introduce a metric $G$ on $\overline{M}$ by

$$G = e^{st}(\pi^*g + \varepsilon dt \otimes dt)$$

(8)

As an induced metric of $g$, we have

$$G((X, 0), (Y, 0)) = g(X, Y) \quad (t = 0)$$

(9)

For any vector fields $\overline{X} = (X, f \frac{d}{dt}), \overline{Y} = (Y, h \frac{d}{dt})$ on $\overline{M}$, we obtain from (7)(8)

$$G(\overline{X}, \overline{Y}) = e^{st}(g(X, Y) + \varepsilon f h)$$

(10)

$$G(J\overline{X}, \overline{Y}) = e^{st}(g(\phi X, Y) - \varepsilon f \eta(Y) + \varepsilon h \eta(X))$$

(11)

$$G(\overline{X}, J\overline{Y}) = e^{st}(g(X, \phi Y) - \varepsilon h \eta(X) + \varepsilon f \eta(Y))$$

(12)

$$G(J\overline{X}, J\overline{Y}) = G((\phi X - f \xi, \eta(X) \frac{d}{dt}), (\phi Y - h \xi, \eta(Y) \frac{d}{dt}))$$

$$= e^{st}(g(\phi X, \phi Y) + \varepsilon f h + \varepsilon \eta(X) \eta(Y))$$

(13)

From (10)-(13), we see

$$G(\overline{X}, J\overline{Y}) = -G(J\overline{X}, \overline{Y}), \quad G(J\overline{X}, J\overline{Y}) = G(\overline{X}, \overline{Y})$$

Thus $G$ is a Hermitian metric on $\overline{M}$.

Define a 2–form on $\overline{M}$ by

$$\Phi = e^{st}(\pi^*d\eta + \varepsilon dt \wedge (\pi^*\eta))$$

(14)

Using $\pi^* \circ d = d \circ \pi^*$, we get

$$d\Phi = e^{st}dt \wedge (\pi^*d\eta + \varepsilon dt \wedge (\pi^*\eta)) +$$

$$e^{st}[\pi^*d^2\eta + \varepsilon d^2t \wedge (\pi^*\eta) - \varepsilon dt \wedge \pi^*d\eta] = 0$$

(15)

Therefore, $\phi$ is a closed 2–form on $\overline{M}$, by a direct computation, we get

$$\Phi(\overline{X}, \overline{Y}) = \Phi((X, f \frac{d}{dt}), (Y, h \frac{d}{dt}))$$

$$= e^{st}(d\eta(X, Y) + \varepsilon(dt \wedge \pi^*\eta) \langle \overline{X}, \overline{Y} \rangle)$$

$$= e^{st}(d\eta(X, Y) + \varepsilon f \eta(Y) - \varepsilon h \eta(X))$$

(16)
From (12) and (16) we see that
\[ \Phi(\overline{X}, \overline{Y}) = G(\overline{X}, J\overline{Y}) \]  
(17)
Then from (15) and (17), we know the Φ defined by (14) is the closed fundamental 2–form, thus the G defined by (8) is an indefinite kaehlerian metric on \( \overline{M}^{[2]} \) and hence \( \overline{M} = M \times R \) is an indefinite kaehlerian manifold.

3. THE PROOF OF THEOREM 1.2

First of all, we state some results which we shall need later.

**Lemmas 3.1.** (see [1] p. 548). An \((e)\)-almost contact metric structure \((\phi, \xi, \eta, g)\) is \((e)\)-sasakian if and only if
\[ (\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y)X, \quad \forall X, Y \in \Gamma(TM) \]  
(18)
Where \( \nabla \) is the Levi–civita connection with respect to \( g \).
If we replace \( Y \) by \( \xi \) in (18) and from (1) (2) we get
\[ \nabla_X \xi = -e\phi X, \quad \forall X \in \Gamma(TM) \]  
(19)
Because
\[ (L_\xi g)(X, Y) = \xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y]) \]
\[ = \xi g(X, Y) - g(\nabla_\xi X - \nabla_X \xi, Y) - g(X, \nabla_\xi Y - \nabla_Y \xi) \]
\[ = (\xi g(X, Y) - g(\nabla_\xi X, Y) - g(X, \nabla_\xi Y + g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \]
\[ = (\nabla_\xi g)(X, Y) + g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \]
\[ = g(-e\phi X, Y) + g(X, -e\phi Y) \]
\[ = -e(g(\phi X, Y) + g(X, \phi Y)) = 0 \quad \forall X, Y \in \Gamma(TM) \]

Then we get

**Proposition 3.1.** The characteristic vector field \( \xi \) on an \((e)\)-sasakian manifold is a killing vector field.

**Lemma 3.2.** ([6] p.265) Let \( M \) be a contact metric manifold with contact metric structure \((\phi, \xi, \eta, g)\). Then \( N^{(3)} \equiv (L_\xi \phi) X \) vanishes if and only if \( \xi \) is a killing vector field with respect to \( g \).

**Proposition 3.2.** Let \( M \) be an \((e)\)-sasakian manifold, then the sectional curvature for plane sections containing \( \xi \) are equal to 1 or -1 at every point on \( M \).

**Proof.** Let \( X \) be an unit vector field on \( M \) and \( X \perp \xi \), then from (19) we have
\[ R(\xi, X)\xi = \nabla_\xi \nabla_X \xi - \nabla_X \nabla_\xi \xi - \nabla_{[\xi, X]} \xi \]
\[ = -e \nabla_\xi (\phi X) + \phi(\xi) \]
\[ = -e(\nabla_\xi (\phi X) - \phi(\nabla_\xi X - \nabla_X \xi) \]
\[ = -e((\nabla_\xi \phi) X + \phi(\nabla_X \xi)) \]
From Lemma 3.1, we get
\[ (\nabla_\xi \phi) X = g(\xi, X)\xi - \eta(X)\xi = 0 \]
thus we have
\[ R(\xi, X)\xi = -\varepsilon\phi(\nabla_X\xi) = \phi^2 X = -X \quad \text{then} \]
\[ g(R(\xi, X)X, \xi) = -g(R(\xi, X)\xi, X) = \pm 1 \]

From (18) and (19), let any \( X, Y \in \Gamma(TM) \) and \( X, Y \perp \xi \) we have
\[
R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X, Y]} \xi \\
= \nabla_X (-\varepsilon\phi Y) - \nabla_Y (-\varepsilon\phi X) + \varepsilon\phi [X, Y] \\
= \varepsilon((\nabla_Y \phi)X - (\nabla_X \phi)Y) \\
= \varepsilon(g(X, Y)\xi - \varepsilon\eta(X)Y - g(X, Y)\xi + \varepsilon\eta(Y)X) \\
= \eta(Y)X - \eta(X)Y = 0
\]

Then, by Proposition 3.1; 3.2, we get the necessary condition of Theorem 2.

Conversely, first, we define a 1-form \( \eta \) and a tensor field of type (1.1) by
\[ \eta(X) = g(X, \xi) \quad \phi X = -\nabla_X \xi \]

We know from [4] \( (\phi, \xi, \eta, g) \) be an almost contact metric structure, satisfying
\[ \phi^2 = -I + \eta \otimes \xi, \quad g(X, \phi Y) = d\eta(X, Y) \]
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \]

Let \( \bar{\xi} = \varepsilon\xi, \bar{\eta} = \varepsilon\eta, \bar{g} = \varepsilon g, \) then
\[ \bar{\eta}(X) = \varepsilon g(X, \xi), \quad \phi X = -\varepsilon \nabla_X \bar{\xi} \]
\[ \phi^2 = -I + \bar{\eta} \otimes \bar{\xi}, \quad g(X, \phi Y) = d\bar{\eta}(X, Y) \]
\[ g(\phi X, \phi Y) = \bar{g}(X, Y) - \varepsilon \eta(X)\eta(Y) \]

Thus \( (\phi, \xi, \eta, g) \) be an \((\varepsilon)\)-contact metric structure.

Now we show that \( N^{(1)} = 0 \), from condition (3) of Theorem 2, we obtain
\[ (\nabla_X \phi)Y = (\nabla_Y \phi)X, \quad \forall X, Y \perp \xi, \quad \text{thus} \]
\[ N_\phi(X, Y) = [\phi, \phi](X, Y) \]
\[ = (\nabla_{\phi X} \phi)Y - (\nabla_{\phi Y} \phi)X + \phi((\nabla_Y \phi)X - (\nabla_X \phi)Y) \]
\[ = (\nabla_{\phi X} \phi)Y - (\nabla_{\phi Y} \phi)X \quad \forall X, Y \perp \xi \]

By using Lemma 3.1, we get
\[ N_\phi(X, Y) = -2\bar{g}(X, \phi Y)\bar{\xi} \]

then
\[ N^{(1)}(X, Y) = N_\phi(X, Y) + 2\bar{g}(X, \phi Y)\bar{\xi} = 0 \]
If $X \perp \xi$, we have by Lemma 3.2

$$N^{(1)}(X, \xi) = N_{\phi}(X, \xi) = \varepsilon \phi(L_{\xi} \phi)X = 0$$

Thus, for any vector field $X, Y$ on $M$, $N^{(1)}(X, Y) = 0$

Hence, the $\varepsilon$–contact metric structure $(\phi, \xi, \eta, g)$ is normal, that is, $M$ is an $\varepsilon$–sasakian manifold with an $(\varepsilon)$–sasakian structure $(\phi, \xi, \eta, g)$.

Theorem 2 can be improved.

**THEOREM 2'**. In order that a $(2n + 1)$–dimensional Riemannian manifold $M$ is $(\varepsilon)$–sasakian manifold, it is necessary and sufficient that the following two conditions are satisfied

1. $M$ admits a unit killing vector field $\xi$
2. $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$ \quad $\forall X, Y \in \Gamma(TM)$

References


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