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ON THE CAUCHY PROBLEM FOR A DEGENERATE PARABOLIC DIFFERENTIAL EQUATION

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ABSTRACT. The aim of this work is to prove the existence and the uniqueness of the solution of a degenerate parabolic equation. This is done using H. Tanabe and P.E. Sobolevskii theory.

KEY WORDS AND PHRASES: Cauchy problem-Degenerate parabolic
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1- INTRODUCTION

We are concerned with the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} - A(x,t,D)u = f(x,t)\ , \quad (x,t) \in R^n x[0,T]\ , \tag{1.1}$$

with the initial data

$$u(x,0) = u_0(x) \tag{1.2}$$

Here we take the operator A(x,t,D) in the form

$$A(x,t,D) = \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x,t)\frac{\partial}{\partial x_k}\right) - \sum_{j=1}^{n} b_j(x,t)\frac{\partial}{\partial x_j} - C(x,t) \tag{1.3}$$

Assume that $\left(a_{jk}(x,t)\right) 1 \le j,k \le n$, $b_j(x,t)$ and $C(x,t)$ are real-valued smooth functions in x and that they are Hölder continuous in t . Moreover $\left(a_{jk}(x,t)\right)$ is assumed to be symmetric and to satisfy the following condition

$$Re \sum_{j,k=1}^{n} a_{jk}(x,t)\,\xi_j\,\xi_k \ge 0\ , \qquad \xi \in R^n\ . \tag{1.4}$$

Assume also that $f(x,t)$ satisfies, for some $\sigma \in (0,1]$

$$|f(x,t) - f(x,\tau)| \le c|t-\tau|^{\sigma} \tag{1.5}$$

for all t, $\tau \in [0,T]$, where c is a positive constant.

Historically, O.A. Oleinik has studied this problem [4]. Her method was elliptic regularization.

In [1] A. El-Fiky also studied non degenerate p-parabolic systems. Also K. Igari [5] has studied this problem by using Friedrichs mollifier.

On the other hand, H. Tanabe [3] and P.E. Sobolevskii [2] have considered the following evolution equation

$$(p)\quad \begin{cases} \dfrac{dv}{dt} + A(t)v = f(t) \\ v(0) = v_o \end{cases}$$

and the following conditions:

1) A is a linear closed operator acting on a Banach space E and its domain of definition D is dense and independent of t.

2) The operator $(\lambda I + A)$ has a bounded inverse satisfying

$$\left\| (\lambda I + A)^{-1} \right\| \le \frac{c_1}{|\lambda| + 1}$$

for any λ with Re $\lambda \ge \beta > 0$, where c_1 and β are positive constants.

3) There exists a positive constant c_2 such that, for some $\sigma \in (0,1]$

$$\| (A(t) - A(\tau))A_\beta^{-1}(s) \| \le c_2 |t - \tau|^\sigma$$

holds for some t, τ, s $\in$ [0,T], where $A_\beta(s) = A(s) + \beta I$.

4) The function f(t) satisfies the following Hölder condition

$$\left\| f(t) - f(\tau) \right\| \le c_3 \left| t - \tau \right|^\sigma$$

where c_3 is a positive constant.

They proved that for any $v_0 \in E$, there exists a unique solution $v(x,t)$ for (p) which is continuous for all $t \in (0,T]$ and continuously differentiable for $t>0$. In case $v_0 \in D(A)$ the solution is continuously differentiable for $t=0$ also.

In this article we shall show that the result of H. Tanabe and P.E. Sobolevskii can be applied to problem (1.1) - (1.2). Our goal is to show that the operator A(x,t;D) which is defined in (1.3) satisfies conditions 1), 2) and 3) mentioned above.

2. PROPOSITIONS AND THEOREM

In this section we state and prove two propositions from which our main theorem follows.

Proposition 1. Take the domain of definition D(A) of the operator A as follows:

$$D(A) = \left\{ u; u \in L^2, Au \in L^2 \right\} \tag{2.1}$$

Then, for large λ, $(\lambda I - A)$ defines a one-to-one surjective mapping of $D(A)$ onto L^2. Moreover there exists a constant α such that

$$\left\| (\lambda I - A)^{-1} \right\| \leq \frac{1}{\lambda - \alpha} \quad \text{for any } \lambda > \alpha, \tag{2.2}$$

Proof. For any $u \in D(A)$ it holds that

$$\| (\lambda I - A) u \| \geq (\lambda^2 - const.\lambda) \| u \|^2 + \| A u \|^2 \tag{2.3}$$

Indeed

$$\begin{aligned} \left\| (\lambda I - A) u \right\|^2 &= \big((\lambda I - A) u, (\lambda I - A) u\big) \\ &= \lambda^2 \| u \|^2 + \| A u \|^2 - 2 \lambda Re (A u, u) \end{aligned} \tag{2.4}$$

Using the condition (1.4), we have

$$2 Re \left(\frac{\partial}{\partial x_j} (a_{jk} \frac{\partial}{\partial x_k}) u, u \right) = -2 \left(a_{jk} \frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_j} \right) \leq 0 \tag{2.5}$$

Similar arguments can be applied to the remaining two terms of the operator A, under the condition that C is uniformly bounded. Hence we obtain (2.3).

The inequality (2.3) shows that, for large λ, $(\lambda I - A)$ defines a one-to-one closed mapping of $D(A)$ into L^2. Therefore we have only to show that the image $(\lambda I - A)\, D(A)$ is dense in L^2. We show this by contradiction. Assume $(\lambda I - A)\, D(A)$ is not dense in L^2. There exists $\psi(\neq 0)$ in L^2 such that

$$\big((\lambda I - A) u, \psi \big) = 0 \qquad \text{for every } u \in D(A).$$

Hence,as D(A) is dense in L^2,

$$(\lambda I - A^*)\, \psi = 0, \tag{2.6}$$

where A^* is the formal adjoint of A.

Since $\psi \in L^2$,(2.6) shows $A^*\psi \in L^2$. If we note that A^* satisfies the same conditions as A, we can use the inequality (2.3) to obtain

$$0 = \left\| (\lambda I - A^*) \psi \right\|^2 \geq (\lambda^2 - const.\lambda) \| \psi \|^2 \tag{2.7}$$

For large λ, this inequality requires the $\psi = 0$. This is contradictory to our assumption $\psi \neq 0$.

Thus the proof is complete

Proposition 2. Assume all the coefficients in (1.1) are smooth in x and Hölder continuous in t. Then

$$\| [A(t) - A(\tau)] A_\beta^{-1}(s) \| \leq c |t - \tau|^\sigma$$

holds for any $t, \tau, s \in (0,T]$.

Proof. For any $\beta > \alpha$ and from proposition 1, $A_\beta(s)$ is a one-to-one linear mapping from $D(A)$ onto L^2. Moreover, it satisfies.

$$\| A_\beta (x, s, d) u \| \geq c_4 \| u \| \tag{2.8}$$

where c_4 is a positive constant. This implies that

$$\|V\| \geq c_4 \|A_\beta^{-1}(x,s,D)V\|$$

Since all the coefficients appearing in (1.1) are assumed to be smooth in x and Hölder continuous in t. So, we have

$$\begin{aligned} &\| \left[A(x,t,D) - A(x,\tau,D)\right] A_\beta^{-1}(x,s,D)\, V \| \\ &\leq c_2 \,|t-\tau|^\sigma \,\| A_\beta^{-1}(x,s,D) V \| \\ &\leq c_2 c_4^{-1} \,|t-\tau|^\sigma \,\|V\|\,. \end{aligned}$$

Thus the proof is complete

The above propositions show that all condition of H. Tanabe and P.E. Sobolevekii are satisfied. Therefore, we have the following theorem.

THEOREM: For any initial data $u_0 \epsilon L_2$ and any right-hand side f(t) satisfying Hölder condition (1.5), there exists a unique solution u(x,t) for the Cauchy problem (1.1)-(1.2) belonging to the space $C_t^o([0,T],L^2) \cap C_t^1([0,T], L^2)$.

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