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AN APPLICATION OF FIXED POINT THEOREMS IN BEST APPROXIMATION THEORY

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ABSTRACT. In this paper, we give an application of Jungck's fixed point theorem to best approximation theory, which extends the results of Singh and Sahab et al.

KEY WORDS AND PHRASES: Contractive operator, best approximant, compatible mappings, fixed point.

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Let X be a normed linear space. A mapping $T : X \to X$ is said to be *contractive* on X (resp., on a subset C of X) if $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in X (resp., C). The set of fixed points of T on X is denoted by $F(T)$. If $\bar{x}$ is a point of X, then for $0 < a \leq 1$, we define the set D_a of best (C, a)-approximants to $\bar{x}$ consists of the points y in C such that

$$a\|y - \bar{x}\| = \inf\{\|z - \bar{x}\| : z \in C\}.$$

Let D denote the set of best C-approximants to $\bar{x}$. For $a = 1$, our definition reduces to the set D of best C-approximants to $\bar{x}$. A subset C of X is said to be *starshaped* with respect to a point $q \in C$ if, for all x in C and all $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)q \in C$. The point p is called the *star-centre* of C. A convex set is starshaped with respect to each of its points, but not conversely. For an example, the set $C = \{0\} \times [0, 1] \cup [1, 0] \times \{0\}$ is starshaped with respect to $(0, 0) \in C$ as the star-centre of C, but it is not convex.

In this paper, we give an application of Jungck's fixed point theorem to best approximation theory, which extends the results of Sahab et al. [9] and Singh [10].

By relaxing the linearity of the operator T and the convexity of D in the original statement of Brosowski [1], Singh [10] proved the following:

Theorem 1. Let C be a T-invariant subset of a normed linear space X. Let $T : C \to C$ be a contractive operator on C and let $\bar{x} \in F(T)$. If $D \subseteq X$ is nonempty, compact and starshaped, then $D \cap F(T) \neq \emptyset$.

In the subsequent paper [11], Singh observed that only the nonexpansiveness of T on $D' = D \cup \{\bar{x}\}$ is necessary. Further, Hicks and Humphries [4] have shown that the assumption $T : C \to C$ can be weakened to the condition $T : \partial C \to C$ if $y \in C$, i.e., $y \in D$ is not necessarily in the interior of C, where ∂C denotes the boundary of C.

Recently, Sahab, Khan and Sessa [9] generalized Theorem 1 as in the following:

Theorem 2. Let X be a Banach space. Let $T, I : X \to X$ be operators and C be a subset of X such that $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(I)$. Further, suppose that T and I satisfy

$$\|Tx - Ty\| \leq \|Ix - Iy\| \tag{1}$$

for all x, y in D', I is linear, continuous on D and $ITx = TIx$ for all x in D. If D is nonempty, compact and starshaped with respect to a point $q \in F(I)$ and $I(D) = D$, then $D \cap F(T) \cap F(I) \neq \emptyset$.

Recall that two self-maps I and T of a metric space (X, d) with $d(x, y) = \|x - y\|$ for all $x, y \in X$ are said to be *compatible* on X if

$$\lim_{n\to\infty} d(ITx_n, TIx_n) (= \lim_{n\to\infty} \|ITx_n - TIx_n\|) = 0$$

whenever there is a sequence $\{x_n\}$ in X such that Tx_n, $Ix_n \to t$, as $n \to \infty$, for some t in X ([6]-[8]). We shall use N to denote the set of positive integers and $Cl(S)$ to denote the closure of a set S.

For our main theorem, we need the following:

Proposition 3. [8] Let T and I be compatible self-maps of a metric space (X, d) with I being continuous. Suppose that there exist real numbers $r > 0$ and $a \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq rd(Ix, Iy) + a \max\{d(Tx, Ix), d(Ty, Iy)\}.$$

Then $Tw = Iw$ for some $w \in X$ if and only if $A = \cap\{Cl(T(K_n)) : n \in N\} \neq \emptyset$, where for each $n \in N$

$$K_n = \{x \in X : d(Tx, Ix) \leq \frac{1}{n}\}.$$

On the other hand, using this proposition, Jungck [8] proved the following:

Theorem 4. Let I and T be compatible self-maps of a closed convex subset C of a Banach space X. Suppose that I is continuous and linear with $T(C) \subseteq I(C)$. If there exists an $a \in (0, 1)$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \leq a\|Ix - Iy\| + (1 - a) \max\{\|Tx - Ix\|, \|Ty - Iy\|\}, \tag{2}$$

then I and T have a unique common fixed point in C.

By using this theorem, we extend Theorem 2 as in the following:

Theorem 5. Let X be a Banach space. Let $T, I : X \to X$ be operators and C be a subset of X such that $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(I)$. Further, suppose that $T$and I satisfy (2) for all x, y in $D'_a = D_a \cup \{\bar{x}\} \cup E$, where $E = \{q \in X : Ix_n, Tx_n \to q,\ \{x_n\} \subset D_a\}$, $0 < a < 1$, I is linear, continuous on D_a and T, I are compatible in D_a. If D_a is nonempty, compact and convex, and $I(D_a) = D_a$, then $D_a \cap F(T) \cap F(I) \neq \emptyset$.

Proof. Let $y \in D_a$ and hence Iy is in D_a since $I(D_a) = D_a$. Further, if $y \in \partial C$, then Ty is in C since $T(\partial C) \subseteq C$. From (2), it follows that

$$\begin{aligned}
\|Ty - \bar{x}\| &= \|Ty - T\bar{x}\| \\
&\leq a\|Iy - I\bar{x}\| + (1 - a) \max\{\|Ty - Iy\|, \|T\bar{x} - I\bar{x}\|\} \\
&\leq a\|Iy - \bar{x}\| + (1 - a)(\|Ty - \bar{x}\| + \|Iy - \bar{x}\|),
\end{aligned}$$

which implies $a\|Ty - \bar{x}\| \leq \|Iy - \bar{x}\|$ and so Ty is in D_a. Thus T maps D_a into itself.

By hypothesis, we have $\bar{x} = T\bar{x} = I\bar{x}$. Then Proposition 3 implies that

$$A = \cap\{Cl(T(K_n)) : n \in N\} \neq \emptyset.$$

Suppose that $w \in A$. Then for each $n \in N$, there exists $y_n \in T(K_n)$ such that $d(w, y_n) < 1/n$. Consequently, for such n, we can and do choose $x_n \in K_n$ such that $d(w, Tx_n) < 1/n$ and so $Tx_n \to w$. But since $x_n \in K_n$, $d(Tx_n, Ix_n) < 1/n$ and therefore $Ix_n \to w$. Thus we have

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = w. \tag{3}$$

Therefore, for a sequence $\{x_n\}$ in D_a the existence of (3) is guaranteed whenever $D_a \subset K_n$. Moreover, $w \in E$. Since I and T are compatible and I is continuous, we have $\lim_{n\to\infty} TIx_n = Iw$ and $\lim_{n\to\infty} I^2x_n = Iw$. By (2), we have

$$\|TIx_n - \bar{x}\| = \|TIx_n - T\bar{x}\| \le a\|I^2x_n - I\bar{x}\| + (1-a)\max\{\|TIx_n - I^2x_n\|, \|T\bar{x} - I\bar{x}\|\},$$

which implies, as $n \to \infty$,

$$\|Iw - \bar{x}\| \le a\|Iw - \bar{x}\|.$$

Hence $Iw = \bar{x}$. By (2) again, we have

$$\|Tw - \bar{x}\| = \|Tw - T\bar{x}\| \le a\|Iw - I\bar{x}\| + (1-a)\max\{\|Tw - Iw\|, \|T\bar{x} - I\bar{x}\|\},$$

which gives $\|Tw - \bar{x}\| \le (1-a)\|Tw - \bar{x}\|$, and so $Tw = \bar{x}$.

Next, we consider

$$\|Tw - Tx_n\| \le a\|Iw - Ix_n\| \qquad + (1-a)\max\{\|Tw - Iw\|, \|Tx_n - Ix_n\|\},$$

which gives $\|\bar{x} - w\| \le a\|\bar{x} - w\|$ as $n \to \infty$, and so $\bar{x} = w$, i.e., $w = Iw = Tw$. By Theorem 4, w must be unique. Hence $E = \{w\}$. Then $D_a^* = D_a \cup \{w\} = D_a'$

Let $\{k_n\}$ be a monotonically non-decreasing sequence of real numbers such that $0 \le k_n < 1$ and $\overline{\lim}_{n\to\infty} k_n = 1$. Let $\{x_j\}$ be a sequence in D_a' satisfying (3). For each $n \in N$, define a mapping $T_n : D_a' \to D_a'$ by

$$T_nx_j = k_nTx_j + (1-k_n)p. \tag{4}$$

It is possible to define such a mapping T_n for each $n \in N$ since D_a' is starshaped with respect to $p \in F(I)$.

Since I is linear, we have

$$T_nIx_j = k_nTIx_j + (1-k_n)p, \quad IT_nx_j = k_nITx_j + (1-k_n)p.$$

By compatibility of I and T, we have for each $n \in N$,

$$\begin{aligned} 0 &\le \lim_{j\to\infty} \|T_nIx_j - IT_nx_j\| \\ &\le k_n \lim_{j\to\infty} \|TIx_j - ITx_j\| + \lim_{j\to\infty} (1-k_n)\|p - p\| \\ &= 0 \end{aligned}$$

and so

$$\lim_{j\to\infty} \|T_nIx_j - IT_nx_j\| = 0$$

whenever $\lim_{j\to\infty} Ix_j = \lim_{j\to\infty} T_nx_j = w$ since we have

$$\begin{aligned} \lim_{j\to\infty} T_nx_j &= k_n \lim_{j\to\infty} Tx_j + (1-k_n)w \\ &= k_nw + (1-k_n)w \\ &= w. \end{aligned}$$

Thus, I and T_n are compatible on D_a' for each n and $T_n(D_a') \subset D_a' = I(D_a')$.

On the other hand, by (2), for all $x, y \in D_a'$, we have, for all $j \ge n$ and n fixed,

$$\begin{aligned} \|T_nx - T_ny\| = k_n\|Tx - Ty\| &\le k_j\|Tx - Ty\| < \|Tx - Ty\| \\ &\le a\|Ix - Iy\| + (1-a)\max\{\|Tx - Ix\|, \|Ty - Iy\|\} \\ &\le a\|Ix - Iy\| + (1-a)\max\{\|Tx - T_nx\| + \|T_nx - Ix\|, \\ &\qquad \|Ty - T_ny\| + \|T_ny - Iy\|\} \\ &\le a\|Ix - Iy\| + (1-a)\max\{(1-k_n)\|Tx - p\| + \|T_nx - Ix\|, \\ &\qquad (1-k_n)\|Ty - p\| + \|T_ny - Iy\|\}. \end{aligned}$$

Hence for all $j \geq n$, we have

$$\begin{aligned}\|T_n x - T_n y\| < a\|Ix - Iy\| + (1-a)\max\{(1-k_j)\|Tx - p\| \\ + \|T_n x - Ix\|, (1-k_j)\|Ty - p\| + \|T_n y - Iy\|\}\end{aligned} \tag{5}$$

Thus, since $\overline{\lim}_{j\to\infty} k_j = 1$, from (5), for every $n \in N$, we have

$$\begin{aligned}\|T_n x - T_n y\| &= \overline{\lim_{j\to\infty}}\, a\|T_n x - T_n y\| \\ &< \overline{\lim_{j\to\infty}}\,[a\|Ix - Iy\| + (1-a)\max\{(1-k_j)\|Tx - p\| \\ &\quad + \|T_n x - Ix\|, (1-k_j)\|Ty - p\| + \|T_n y - Iy\|\}],\end{aligned}$$

which implies

$$\|T_n x - T_n y\| = a\|Ix - Iy\| + (1-a)\max\{\|T_n x - Ix\|, \|T_n y - Iy\|\}$$

for all $x, y \in D'_a$. Therefore, by Theorem 4, for every $n \in N$, T_n and I have a unique common fixed point x_n in D'_a, i.e., for every $n \in N$, we have

$$F(T_n) \cap F(I) = \{x_n\}.$$

Now, the compactness of D_a ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ which converges to a point z in D_a. Since

$$x_{n_i} = T_{n_i} x_{n_i} = k_{n_i} T x_{n_i} + (1 - k_{n_i}) q \tag{6}$$

and T is continuous, we have, as $i \to \infty$ in (6), $z = Tz$, i.e., $z \in D_a \cap F(T)$.

Further, the continuity of I implies that

$$Iz = I(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} I x_{n_i} = \lim_{i\to\infty} x_{n_i} = z,$$

i.e., $z \in F(I)$. Therefore, we have $z \in D_a \cap F(T) \cap F(I)$ and so

$$D_a \cap F(T) \cap F(I) \neq \emptyset.$$

This completes the proof.

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