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ASYMPTOTIC THEORY FOR A CRITICAL CASE FOR A GENERAL FOURTH-ORDER DIFFERENTIAL EQUATION

A.S.A. AL-HAMMADI
Department of Mathematics
College of Science
University of Bahrain
P.O. Box 32088
Isa Town, BAHRAIN

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ABSTRACT. In this paper we identify a relation between the coefficients that represents a critical case for general fourth-order equations. We obtained the forms of solutions under this critical case

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1. INTRODUCTION

We consider the general fourth-order differential equation

$$(p_0y'')'' + (p_1y')' + \frac{1}{2}\sum_{j=0}^{1}\left[\left\{q_{2-j}y^{(j+1)}\right\} + \left\{q_{2-j}y^{(j+1)}\right\}^{(j)}\right] + p_2y = 0 \tag{1.1}$$

where x is the independent variable and the prime denotes d/dx. The functions $p_i(x)(0 \leq i \leq 2)$ and $q_i(x)(i = 1, 2)$ are defined on an interval $[a, \infty)$ and are not necessarily real-valued and are all nowhere zero in this interval. Our aim is to identify relations between the coefficients that represent a critical case for (1.1) and to obtain the asymptotic forms of our linearly independent solutions under this case. Al-Hammadi [1] considered (1.1) with the case where p_0 and p_2 are the dominate coefficients and we give a complete analysis for this case Similar fourth-order equations to (1.1) have been considered previously by Walker [2, 3] and Al-Hammadi [4]. Eastham [5] considered a critical case for (1 1) with $p_1 = q_2 = 0$ and showed that this case represents a borderline between situations where all solutions have a certain exponential character as $x \to \infty$ and where only two solutions have this character.

The critical case for (1.1) that has been referred, is given by:

$$\frac{q_i'}{q_i} \sim \text{const.}\ \frac{p_2}{q_2}\ (i = 1, 2), \quad \frac{\left(p_1 q_1^{-1/2}\right)'}{p_1 q_1^{-1/2}} \sim \text{const}\ \frac{p_2}{q_2}. \tag{1.2}$$

We shall use the recent asymptotic theorem of Eastham [6, section 2] to obtain the solutions of (1.1) under the above case. The main theorem for (1.1) is given in section 4 with discussion in section 5.

2. A TRANSFORMATION OF THE DIFFERENTIAL EQUATION

We write (1.1) in the standard way [7] as a first order system

$$Y' = AY, \tag{2.1}$$

where the first component of Y is y and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2}q_1p_0^{-1} & p_0^{-1} & 0 \\ -\frac{1}{2}q_2 & -p_1+\frac{1}{4}q_1^2p_0^{-1} & -\frac{1}{2}p_0^{-1}q_1 & 1 \\ -p_2 & -\frac{1}{2}q_2 & 0 & 0 \end{bmatrix}. \tag{2.2}$$

As in [4], we express A in its diagonal form

$$T^{-1}AT = \Lambda, \tag{2.3}$$

and we therefore require the eigenvalues λ_j and eigenvectors $v_j (1 \le j \le 4)$ of A.

The characteristic equation of A is given by

$$p_0\lambda^4 + q_1\lambda^3 + p_1\lambda^2 + q_2\lambda + p_2 = 0. \tag{2.4}$$

An eigenvector v_j of A corresponding to λ_j is

$$v_j = \left(1, \lambda_j, p_0\lambda_j^2 + \frac{1}{2}q_1\lambda_j, -\frac{1}{2}q_2 - p_2\lambda_j^{-1}\right)^t \tag{2.5}$$

where the superscript t denotes the transpose. We assume at this stage that the λ_j are distinct, and we define the matrix T in (2.3) by

$$T = (v_1\ v_2\ v_3\ v_4). \tag{2.6}$$

Now from (2.2) we note that EA coincides with its own transpose, where

$$E = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.7}$$

Hence, by [8, section 2(i)], the v_j have the orthogonality property

$$(Ev_k)^t v_j = 0 \quad (k \neq j). \tag{2.8}$$

We define the scalars $m_j (1 \le j \le 4)$ by

$$m_j = (Ev_j)^t v_j, \tag{2.9}$$

and the row vectors

$$r_j = (Ev_j)^t. \tag{2.10}$$

Hence, by [8, section 2]

$$T^{-1} = \begin{bmatrix} m_1^{-1}r_1 \\ m_2^{-1}r_2 \\ m_3^{-1}r_3 \\ m_4^{-1}r_4 \end{bmatrix}, \tag{2.11}$$

and

$$m_j = 4p_0\lambda_j^3 + 3q_1\lambda_j^2 + 2p_2\lambda_j + q_2. \tag{2.12}$$

Now we define the matrix U by

$$U = (v_1\ v_2\ v_3\ \epsilon_1\ v_4) = TK, \tag{2.13}$$

where

$$\epsilon_1 = \frac{p_0 p_1}{q_1^2}, \tag{2.14}$$

the matrix K is given by

$$K = dg(1, 1, 1, \epsilon_1). \tag{2.15}$$

By (2.3) and (2.13), the transformation

$$Y = UZ \tag{2.16}$$

takes (2.1) into

$$Z' = (\Lambda - U^{-1}U')Z. \tag{2.17}$$

Now by (2.13),

$$U^{-1}U' = K^{-1}T^{-1}T'K + K^{-1}K', \tag{2.18}$$

where

$$K^{-1}K' = dg(0, 0, 0, \epsilon_1^{-1}\epsilon_1'), \tag{2.19}$$

and we use (2.15).

Now we write

$$U^{-1}U' = \phi_{ij} \quad (1 \le i, j \le 4), \tag{2.20}$$

and

$$T^{-1}T' = \psi_{ij} \quad (1 \le i, j \le 4), \tag{2.21}$$

then by (2.18) to (2.21), we have

$$\phi_{ij} = \psi_{ij}, \qquad (1 \le i, j \le 3), \tag{2.22}$$

$$\phi_{44} = \psi_{44} + \epsilon_1^{-1}\epsilon_1', \tag{2.23}$$

$$\phi_{i4} = \psi_{i4}\epsilon_1 \qquad (1 \le i \le 3), \tag{2.24}$$

$$\phi_j = \epsilon_1^{-1}\psi_{4j} \qquad (1 \le j \le 3). \tag{2.25}$$

Now to work out $\phi_{ij}(1 \le i, j \le 4)$, it suffices to deal with ψ_{ij} of the matrix $T^{-1}T'$. Thus by (2.6), (2.10), (2.11) and (2.12) we obtain

$$\psi_{ii} = \frac{1}{2}\frac{m_i'}{m_i} \quad (1 \le i \le 4) \tag{2.26}$$

and, for $i \ne j$, $1 \le i, j \le 4$

$$\psi_{ij} = m_i^{-1}\left\{\lambda_j'\left(p_0\lambda_i^2 + \frac{1}{2}q_1\lambda_i\right) + \lambda_i\left(p_0\lambda_j^2 + \frac{1}{2}q_1\lambda_j\right)' - \frac{1}{2}q_2' - (p_2\lambda_j^{-1})'\right\}. \tag{2.27}$$

Now we need to work out (2.26) and (2.27) in some detail in terms of p_0, p_1, p_2, q_1 and q_2 and then (2.22)-(2.25) in order to determine the form of (2.17).

3. THE MATRICES L, $T^{-1}T$ AND $U^{-1}U$

In our analysis, we impose a basic condition on the coefficients, as follows:

(I) $p_i(0 \le i \le 2)$ and $q_i(i = 1, 2)$ are nowhere zero in some interval $[a, \infty)$, and

$$\frac{p_i}{q_{i+1}} = o\left(\frac{q_{i+1}}{p_{i+1}}\right) \quad (i = 0, 1) \quad (x \to \infty) \tag{3.1}$$

and

$$\frac{q_1}{p_1} = o\left(\frac{p_1}{q_2}\right). \tag{3.2}$$

If we write

$$\epsilon_1 = \frac{p_0 p_1}{q_1^2}, \quad \epsilon_2 = \frac{q_1 q_2}{p_1^2}, \quad \epsilon_3 = \frac{p_2 p_1}{q_2^2}, \tag{3.3}$$

then by (3.1) and (3.2) for $(1 \leq i \leq 3)$

$$\epsilon_i = o(1) \quad (x \to \infty). \tag{3.4}$$

Now as in [4], we can solve the characteristic equation (2.4) asymptotically as $x \to \infty$. Using (3.1), (3.2) and (3.3) we obtain the distinct eigenvalues λ_j as

$$\lambda_1 = -\frac{p_2}{q_2}(1 + \delta_1), \tag{3.5}$$

$$\lambda_2 = -\frac{q_2}{p_1}(1 + \delta_2), \tag{3.6}$$

$$\lambda_3 = -\frac{p_1}{q_1}(1 + \delta_3), \tag{3.7}$$

and

$$\lambda_4 = -\frac{q_1}{p_0}(1 + \delta_4), \tag{3.8}$$

where

$$\delta_1 = 0(\epsilon_3), \quad \delta_2 = 0(\epsilon_2) + 0(\epsilon_3), \quad \delta_3 = 0(\epsilon_1) + 0(\epsilon_2), \quad \delta_4 = (\epsilon_1). \tag{3.9}$$

Now by (3.1) and (3.2), the ordering of λ_j is such that

$$\lambda_j = o(\lambda_{j+1}) \quad (x \to \infty, 1 \leq j \leq 3). \tag{3.10}$$

Now we work out $m_j (1 \leq j \leq 4)$ asymptotically as $x \to \infty$, hence by (3.3)-(3.9), (2.12) gives for $(1 \leq j \leq 4)$

$$m_1 = q_2\{1 + 0(\epsilon_3)\}, \tag{3.11}$$

$$m_2 = -q_2\{1 + 0(\epsilon_2) + 0(\epsilon_3)\}, \tag{3.12}$$

$$m_3 = \frac{p_1^2}{q_1}\{1 + 0(\epsilon_1) + 0(\epsilon_2)\}, \tag{3.13}$$

and

$$m_4 = -\frac{q_1^3}{p_0^2}\{1 + 0(\epsilon_1)\}. \tag{3.14}$$

Also on substituting $\lambda_j (j = 1, 2, 3, 4)$ into (2.12) and using (3.5)-(3.8) respectively and differentiating, we obtain

$$m_1' = q_2'\{1 + 0(\epsilon_3)\} + q_2\left\{0(\epsilon_3') + 0(\epsilon_3\delta_1') + 0\left(\epsilon_2'\epsilon_3^2\right) + 0\left(\epsilon_1'\epsilon_2^2\epsilon_3^3\right)\right\}, \tag{3.15}$$

$$m_2' = -q_2'\{1+0(\epsilon_2)+0(\epsilon_3)\}+q_2\{0(\delta_2')+0(\epsilon_2')+0(\epsilon_1'\epsilon_2^2)\}, \tag{3.16}$$

$$m_3' = \left(\frac{p_1^2}{q_1}\right)'\{1+0(\epsilon_1)+0(\epsilon_2)\}+\frac{p_1^2}{q_1}\{0(\delta_3')+0(\epsilon_2')+0(\epsilon_1')\}, \tag{3.17}$$

and

$$m_4' = -\left(\frac{q_1^3}{p_0^2}\right)'\{1+0(\epsilon_2)\}+\frac{q^3}{p_0^2}\{0(\epsilon_2'\epsilon_1^2)+0(\epsilon_1')\}. \tag{3.18}$$

At this stage we also require the following conditions

(II) $\dfrac{p_0'}{p_0}\epsilon_i,\quad \dfrac{p_1'}{p_1}\epsilon_i,\quad \dfrac{q_1'}{q_1}\epsilon_i,\quad \dfrac{q_2'}{q_2}\epsilon_i,\quad \dfrac{p_2'}{p_2}\epsilon_2,\quad \dfrac{p_2'}{p_2}\epsilon_3$ are all

$$L(a,\infty)\quad (1\le i\le 3). \tag{3.19}$$

Further, differentiating (3.3) for $\epsilon_i (1\le i\le 3)$, we obtain

$$\epsilon_1' = 0\left(\frac{p_0'}{p_0}\epsilon_1\right)+0\left(\frac{p_1'}{p_1}\epsilon_1\right)+0\left(\frac{q_1'}{q_1}\epsilon_1\right), \tag{3.20}$$

$$\epsilon_2' = 0\left(\frac{q_1'}{q_1}\epsilon_2\right)+0\left(\frac{q_2'}{q_2}\epsilon_2\right)+0\left(\frac{p_1'}{p_1}\epsilon_2\right), \tag{3.21}$$

and

$$\epsilon_3' = 0\left(\frac{p_2'}{p_2}\epsilon_3\right)+0\left(\frac{p_1'}{p_1}\epsilon_3\right)+0\left(\frac{q_2'}{q_2}\epsilon_3\right). \tag{3.22}$$

For reference shortly, we note on substituting (3.5)-(3.8) into (2.4) and differentiating, we obtain

$$\delta_1' = 0(\epsilon_3')+0(\epsilon_2'\epsilon_3^2)+0(\epsilon_1'\epsilon_3^3\epsilon_2^2), \tag{3.23}$$

$$\delta_2' = 0(\epsilon_2')+0(\epsilon_3')+0(\epsilon_1'\epsilon_3^2), \tag{3.24}$$

$$\delta_3' = 0(\epsilon_1')+0(\epsilon_2')+0(\epsilon_3'\epsilon_2^2), \tag{3.25}$$

and

$$\delta_4' = 0(\epsilon_1')+0(\epsilon_2'\epsilon_1^2)+0(\epsilon_3'\epsilon_1^3\epsilon_2^2). \tag{3.26}$$

Hence by (3.19) and (3.20)-(3.26)

$$\epsilon_j' \text{ and } \delta_j' \text{ are } L(a,\infty). \tag{3.27}$$

For the diagonal elements $\psi_{ii} (1\le j\le 4)$ in (2.26) we can now substitute the estimates (3.11)-(3.18) into (2.26). We obtain

$$\psi_{11} = \frac{1}{2}\frac{q_2'}{q_2}+0\left(\frac{q_2'}{q_2}\epsilon_3\right)+0(\epsilon_3')+0(\epsilon_3\delta_1')+0(\epsilon_2'\epsilon_3^2)+0(\epsilon_1'\epsilon_2^2\epsilon_3^3), \tag{3.28}$$

$$\psi_{22} = \frac{1}{2}\frac{q_2'}{q_2}+0\left(\frac{q_2'}{q_2}\epsilon_2\right)+0\left(\frac{q_2'}{q_2}\epsilon_3\right)+0(\delta_2')+0(\epsilon_2')+0(\epsilon_1'\epsilon_2^2), \tag{3.29}$$

$$\psi_{33} = \frac{1}{2}\left[2\frac{p_1'}{p_1} - \frac{q_1'}{q_1}\right] + 0\left(\frac{p_1'}{p_1}\epsilon_1\right) + 0\left(\frac{p_1'}{p_1}\epsilon_2\right) + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0(\delta_3') + 0(\epsilon_2') + 0(\epsilon_1'), \tag{3.30}$$

$$\psi_{44} = \frac{1}{2}\left[3\frac{q_1'}{q_1} - 2\frac{p_0'}{p_0}\right] + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0(\delta_4') + 0(\epsilon_2'\epsilon_1^2) + 0(\epsilon_1'). \tag{3.31}$$

Now for the non-diagonal elements $\psi_{ij} (i \neq j, 1 \leq i, j \leq 4)$, we consider (2.27). Hence (2.27) gives for $i = 1$ and $j = 2$

$$\psi_{12} = m_1^{-1}\left\{\lambda_2'\left(p_0\lambda_1^2 + \frac{1}{2}q_1\lambda_1\right) + \lambda_1\left(p_0\lambda_2^2 + \frac{1}{2}q_1\lambda_2\right)' - \frac{1}{2}q_2' - (p_2\lambda_2^{-1})'\right\}. \tag{3.32}$$

Now by (3.5), (3.6), (3.3) and (3.11) we have

$$m_1^{-1}\lambda_2'\left(p_0\lambda_1^2 + \frac{1}{2}q_1\lambda_1\right) = \frac{1}{2}\left[\frac{q_2'}{q_2} - \frac{p_1'}{p_1}\right]\epsilon_2\epsilon_3\{1 + 0(\epsilon_3)\} + 0(\epsilon_2\epsilon_3\delta_2'), \tag{3.33}$$

$$m_1^{-1}\lambda_1\left(p_0\lambda_2^2 + \frac{1}{2}q_1\lambda_2\right)' = 0(\epsilon_2\epsilon_3\delta_2') + 0(\epsilon_2^2\epsilon_1\epsilon_3)\left[\frac{p_0'}{p_0} + 2\frac{q_2'}{q_2} - 2\frac{p_1'}{p_1}\right] + 0(\epsilon_2\epsilon_3)\left[\frac{q_1'}{q_1} + \frac{q_2'}{q_2} - \frac{p_1'}{p_1}\right], \tag{3.34}$$

$$-\frac{1}{2}q_2'm_1^{-1} = -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_3\right), \tag{3.35}$$

and

$$m_1^{-1}(p_2\lambda_2^{-1})' = 0\left(\frac{p_2'}{p_2}\epsilon_3\right) + 0\left(\frac{p_1'}{p_1}\epsilon_3\right) + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0(\epsilon_3\delta_2'). \tag{3.36}$$

Hence by (3.33)-(3.36), (3.32) gives

$$\psi_{12} = -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0\left(\frac{p_1'}{p_1}\epsilon_3\right) + 0\left(\frac{p_2'}{p_2}\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_2^2\epsilon_3\right) + 0(\epsilon_3\delta_2') + 0\left(\frac{q_1'}{q_1}\epsilon_2\epsilon_3\right). \tag{3.37}$$

Similar work can be done for the other elements ψ_{ij}, so we obtain

$$\psi_{13} = -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0\left(\frac{p_1'}{p_1}\epsilon_3\right) + 0\left(\frac{q_1'}{q_1}\epsilon_3\right) + 0(\epsilon_3\delta_3') + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_3\right) + 0\left(\frac{p_2'}{p_2}\epsilon_2\epsilon_3\right). \tag{3.38}$$

$$\psi_{14} = -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0\left(\frac{q_1'}{q_1}\epsilon_1^{-1}\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1^{-1}\epsilon_3\right) + 0(\epsilon_1^{-1}\epsilon_3\delta_{4'}') + 0\left(\frac{p_2'}{p_2}\epsilon_1\epsilon_2\epsilon_3\right). \tag{3.39}$$

$$\psi_{21} = -\frac{1}{2}\frac{q_2'}{q_2} + 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0(\delta_1') + 0\left(\epsilon_2\frac{p_2'}{p_2}\right) + 0\left(\epsilon_3\frac{p_2'}{p_2}\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_2^2\epsilon_3^2\right) \tag{3.40}$$

$$\psi_{23} = \left[\frac{1}{2}\frac{q_1'}{q_1} - \frac{p_1'}{p_1} + \frac{1}{2}\frac{q_2'}{q_2}\right] + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0\left(\frac{q_1'}{q_1}\epsilon_3\right) + 0\left(\frac{p_1'}{p_1}\epsilon_1\right) + 0\left(\frac{p_1'}{p_1}\epsilon_2\right) + 0\left(\frac{p_1'}{p_1}\epsilon_3\right) + 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0\left(\frac{q_2'}{q_2}\epsilon_3\right) + 0(\delta_3') + 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0\left(\epsilon_2\epsilon_3\frac{p_2'}{p_2}\right), \tag{3.41}$$

$$\psi_{24} = \epsilon_1^{-1}\left[\frac{1}{2}\frac{q_1'}{q_1} + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0\left(\frac{q_1'}{q_1}\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0\left(\frac{p_0'}{p_0}\epsilon_2\right) + 0\left(\frac{p_0'}{p_0}\epsilon_3\right) + 0(\delta_4') + 0\left(\frac{q_2'}{q_2}\epsilon_1\right) + 0\left(\frac{p_2'}{p_2}\epsilon_1^2\epsilon_2\epsilon_3\right)\right] \tag{3.42}$$

$$\psi_{31} = 0\left(\frac{p_2'}{p_2}\epsilon_2\right) + 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0(\delta_1'\epsilon_2) + 0\left(\frac{q_1'}{q_1}\epsilon_2\epsilon_3\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_2^2\epsilon_3^2\right) \tag{3.43}$$

$$\psi_{32} = 0\left(\frac{q_2'}{q_2}\epsilon_2\right) + 0\left(\frac{p_1'}{p_1}\epsilon_2\right) + 0(\epsilon_2\delta_2') + 0\left(\epsilon_1\epsilon_2^2\frac{p_0'}{p_0}\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0\left(\epsilon_2\epsilon_3\frac{p_2'}{p_2}\right), \tag{3.44}$$

$$\psi_{34} = \epsilon_1^{-1}\left[-\frac{1}{2}\frac{q_1'}{q_1} + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0\left(\frac{p_0'}{p_0}\epsilon_2\right) + 0(\delta_4') + 0\left(\frac{q_1'}{q_1}\epsilon_1\epsilon_2\right) + 0\left(\frac{p_2'}{p_2}\epsilon_1^2\epsilon_2^2\epsilon_3\right)\right] \tag{3.45}$$

$$\psi_{41} = \epsilon_1\left[0\left(\frac{q_1'}{q_1}\epsilon_2\epsilon_3\right) + 0\left(\frac{q_2'}{q_2}\epsilon_1\epsilon_2\right) + 0\left(\frac{p_2'}{p_2}\epsilon_1\epsilon_2\right) + 0(\delta_1'\epsilon_1\epsilon_2) + 0\left(\frac{p_0'}{p_0}\epsilon_1\epsilon_2^2\epsilon_3^2\right)\right] \tag{3.46}$$

$$\psi_{42} = 0\left(\frac{q_2'}{q_2}\epsilon_1\epsilon_2\right) + 0\left(\frac{p_1'}{p_1}\epsilon_1\epsilon_2\right) + 0(\delta_2'\epsilon_1\epsilon_2) + 0\left(\frac{q_1'}{q_1}\epsilon_1\epsilon_2\right) + 0\left(\frac{p_0'}{p_0}\epsilon_1^2\epsilon_2^2\right) + 0\left(\frac{p_2'}{p_2}\epsilon_1^2\epsilon_2\epsilon_3\right), \tag{3.47}$$

$$\psi_{43} = \epsilon_1\left[-\frac{1}{2}\frac{q_1'}{q_1} + 0\left(\frac{p_1'}{p_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_1\right) + 0\left(\frac{q_1'}{q_1}\epsilon_2\right) + 0(\delta_3'\epsilon_1)\right. \left. 0\left(\frac{p_0'}{p_0}\epsilon_1\right) + 0\left(\frac{p_2'}{p_2}\epsilon_1\epsilon_2^2\epsilon_3\right) + 0\left(\frac{q_2'}{q_2}\epsilon_1\epsilon_2\right)\right. \tag{3.48}$$

Now we need to work out (2.22)-(2.25) in order to determine the form (2.17). Now by (3.28)-(3.31) and (3.37)-(3.48), (2.22)-(2.25) will give:

$$\phi_{11} = \frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_1), \qquad \phi_{22} = \frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_2)$$
$$\phi_{33} = \frac{p_1'}{p_1} - \frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_3), \quad \phi_{44} = \frac{p_1'}{p_1} - \frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_4) \tag{3.49}$$

$$
\begin{aligned}
&\phi_{12} = -\frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_5), && && \phi_{13} = -\frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_6)\\
&\phi_{14} = 0(\Delta_7), && && \phi_{21} = -\frac{1}{2}\frac{q_2'}{q_2} + 0(\Delta_8)\\
&\phi_{23} = \frac{1}{2}\left(\frac{q_1'}{q_1} + \frac{q_2'}{q_2}\right) - \frac{p_1'}{p_1} + 0(\Delta_9), && && \phi_{24} = \frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_{10})\\
&\phi_{31} = 0(\Delta_{11}), && \phi_{32} = 0(\Delta_{12}), && \phi_{34} = -\frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_{13})\\
&\phi_{41} = 0(\Delta_{14}), && \phi_{42} = 0(\Delta_{15}), && \phi_{43} = -\frac{1}{2}\frac{q_1'}{q_1} + 0(\Delta_{16}).
\end{aligned}
\tag{3 50}
$$

where

$$\Delta_i \quad \text{is} \quad L(a,\infty) \ (1 \le i \le 16) \tag{3.51}$$

by (3.19) and (3.27).

Now by (3.49)-(3.51), we write the system (2.17) as

$$Z' = (\Lambda + R + S)Z \tag{3.52}$$

where

$$R = \begin{bmatrix} -\eta_1 & \eta_1 & \eta_1 & 0 \\ \eta_1 & -\eta_1 & \eta_2 - \eta_1 & -\eta_3 \\ 0 & 0 & -\eta_2 & \eta_3 \\ 0 & 0 & \eta_3 & -\eta_2 \end{bmatrix} \tag{3.53}$$

with

$$\eta_1 = \frac{1}{2}\frac{q_2'}{q_2}, \quad \eta_2 = \frac{\left(p_1 q_1^{-1/2}\right)'}{p_1 q_1^{-1/2}}, \quad \eta_3 = \frac{1}{2}\frac{q_1'}{q_1}, \tag{3.54}$$

and S is $L(a,\infty)$ by (3.51).

4. THE ASYMPTOTIC FORM OF SOLUTIONS

THEOREM 4.1. Let the coefficients q_1, q_2 and p_1 in (1.1) be $C^{(2)}[a,\infty)$ and let p_0 and p_2 to be $C^{(1)}[a,\infty)$. Let (3.1), (3.2) and (3.19) hold. Let

$$\eta_k = \omega_k \frac{p_2}{q_2}(1 + \psi_k) \tag{4.1}$$

where $\omega_k (1 \le k \le 3)$ are "non-zero" constants and $\psi_k(x) \to 0\,(1 \le k \le 3, x \to \infty)$. Also let

$$\psi_k'(x) \quad \text{is} \quad L(a,\infty) \ (1 \le k \le 3). \tag{4.2}$$

Let

$$\operatorname{Re} I_j(x) (j = 1,2) \quad \text{and} \quad \operatorname{Re}\left[\frac{1}{2}(\lambda_3 + \lambda_4 + \eta_2 + \eta_4 - \lambda_1 - \lambda_2) \pm I_1 \pm I_2\right]$$
be of one sign in $[a,\infty)$ (4.3)

where

$$I_1 = \left[4\eta_1^2 + (\lambda_1 - \lambda_2)^2\right]^{1/2}, \tag{4.4}$$

$$I_2 = \left[4\eta_3^2 + (\lambda_3 - \lambda_4)^2\right]^{1/2}. \tag{4.5}$$

Then (1.1) has solutions

$$y_k \sim q_2^{-1/2}\exp\left(\frac{1}{2}\int_a^x [\lambda_1+\lambda_2+(-1)^{k+1}I_1]\,dt\right),\ (k=1,2) \tag{4.6}$$

$$y_3 \sim q_1^{1/2}p_1^{-1}\exp\left(\frac{1}{2}\int_a^x [\lambda_3+\lambda_4+I_2]\,dt\right), \tag{4.7}$$

$$y_4 = o\left\{q_1^{1/2}p_1^{-1}\exp\left(\frac{1}{2}\int_a^x [\lambda_3+\lambda_4-I_2]\,dt\right)\right\}. \tag{4.8}$$

PROOF. As in [4] we apply Eastham Theorem [6, section 2] to the system (3.52) provided only that Λ and R satisfy the conditions and we shall use (3.53), (3.54), (4.1) and (4.2). We first require that

$$\eta_k = o\{(\lambda_i-\lambda_i)\}\ \ (i \neq j, 1 \leq i,k,j, \leq 4, k \neq 3), \tag{4.9}$$

this being [6, (2.1)] for our system. By (4.1), (3.54), (3.5)-(3.8), this requirement is implied by (3.1) and (3.2).

We also require that

$$\{\eta_k(\lambda_i-\lambda_j)^{-1}\}' \in L(a,\infty)\ \ (1 \leq k \leq 3), \tag{4.10}$$

for $(i \neq j)$ this being [9, (2.2)] for our system. By (4.1), (3.54), (3.5)-(3.8), this requirement is implied by (3.19) and (4.2). Finally we require the eigenvalues $\mu_k (1 \leq k \leq 4)$ of $\Lambda + R$ satisfy the dichotomy condition [10], as in [4], the dichotomy condition holds if

$$\mu_j - \mu_k = f + g\,(j \neq k, 1 \leq j,k \leq 4) \tag{4.11}$$

where f has one sign in $[a,\infty)$ and $g \in L(a,\infty)$ [6, (1.5)]. Now by (2.3) and (3.53)

$$\mu_k = \frac{1}{2}(\lambda_1+\lambda_2-2\eta_1)+\frac{1}{2}(-1)^{k+1}I_1,\ \ (k=1,2) \tag{4.12}$$

$$\mu_k = \frac{1}{2}(\lambda_3+\lambda_4-2\eta_2)+\frac{1}{2}(-1)^{k+1}I_2,\ \ (k=3,4). \tag{4.13}$$

Thus by (4.3), (4.11) holds since (3.52) satisfies all the conditions for the asymptotic result [6, section 2], it follows that as $x \to \infty$, (2.17) has four linearly independent solutions,

$$Z_k(x) = \{e_k + o(1)\}\exp\left(\int_a^x \mu_k(t)\,dt\right), \tag{4.14}$$

where e_k is the coordinate vector with k-th component unity and other components zero. We now transform back to Y by means of (2.13) and (2.16). By taking the first component on each side of (2.16) and making use of (4.12) and (4.13) and carrying out the integration of $-\frac{1}{2}\frac{q_2}{q_2}$ and $\frac{(q_1^{1/2}p_1^{-1})}{q_1^{1/2}p_1^{-1}}$ for $(1 \leq k \leq 4)$ respectively we obtain (4.6), (4.7) and (4.8) after an adjustment of a constant multiple in $y_k (1 \leq k \leq 3)$.

5. DISCUSSION

(i) In the familiar case the coefficients which are covered by Theorem 4.1 are

$$p_i(x) = c_i x^{\alpha_i} (i = 0,1,2,),\quad q_i(x) = c_{i+2}x^{\alpha_{i+2}} (i=1,2)$$

with real constants α_i and $c_i (0 \leq i \leq 4)$. Then the critical case (4.1) is given by

$$\alpha_4 - \alpha_2 = 1. \tag{5.1}$$

The values of $\omega_k (1 \leq k \leq 3)$ in (4.1) are given by

$$\omega_1 = \frac{1}{2}\alpha_4 c_2 c_4^{-1}, \quad \omega_2 = \left(\alpha_1 - \frac{1}{2}\alpha_3\right)c_2 c_4^{-1}, \quad \omega_3 = \frac{1}{2}\alpha_3 c_2 c_4^{-1}, \tag{5.2}$$

where

$$\psi_k(x) = 0\,(1 \le k \le 4). \tag{5.3}$$

(ii) More general coefficients are

$$p_0 = c_0 x^{\alpha_0} e^{-2x^b}, \quad p_1 = c_1 x_1^{\alpha} e^{\frac{1}{2}x^b}, \quad p_2 = c_2 x^{\alpha_2} e^{x^b},$$

$$q_1 = c_3 x^{\alpha_3} e^{-\frac{1}{2}x^b}, \quad q_2 = c_4 x^{\alpha_4} e^{x^b}.$$

with real constants c_i, α_i $(0 \le i \le 4)$ and $b(>0)$. Then the critical case (4.1) is given by

$$\alpha_2 - \alpha_4 = b - 1 \tag{5.4}$$

and the values of $\omega_k (1 \le k \le 4)$ are given by

$$\omega_1 = \frac{1}{2} b c_4 c_{72}^{-1}, \quad \omega_2 = \frac{3}{2}\omega_1, \quad \omega_3 = -\frac{1}{2}\omega_1,$$

with $\psi_1 = \alpha_4 b^{-1} x^{-b}$, $\psi_2 = \frac{4}{3} b^{-1}(\alpha_1 - \frac{1}{2}\alpha_3)x^{-b}$, $\psi_3 = -2\alpha_3 b^{-1} x^{-b}$. Here it is clear that $\psi_{k'}' \in L(a, \infty)$ because $b > 0$.

(iii) We note that in both critical cases (5.1) and (5.4) represent an equation of line in the $\alpha_2\alpha_4$-plane.

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