

## ON DIRICHLET CONVOLUTION METHOD

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**ABSTRACT.** In this paper we have proved limitation theorem for  $(D, h(n))$  summability methods and have shown that it is best possible.

**KEY WORDS AND PHRASES:** Summability, Dirichlet convolution methods

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### 1. INTRODUCTION

In his studies on the prime number theorem, Ingham [1] defined a novel summability method called (I). This was generalized by Segal [2] and he defined the notion of  $(D, h(n))$  summability, where  $h : N \rightarrow R$  denotes a function with  $h(1) = 1$ . We define the "Dirichlet inverse"  $h^*(n)$  of  $h(n)$  by  $\sum_{d|n} h(d)h^*(n/d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$ . A series  $\sum a_n$  is said to be  $(D, h(n))$  summable to  $L$  if and only if

$$n \xrightarrow{\text{lim}} \infty \frac{1}{n} \sum_{v=1}^n v \sum_{d|v} a_d h(v/d) = L. \tag{1.1}$$

Given a series  $\sum a_n$  and a specific  $h(n)$ , define the function

$$D(t) = \frac{1}{t} \sum_{n < t} n \sum_{d|n} a_d h(n/d). \tag{1.2}$$

Since  $D([t]) = \frac{t}{[t]} D(t)$ , it clearly makes no difference to the existence or value of the limit (1.2) whether  $t \rightarrow \infty$  is through real values or integers. Ingham's method corresponds to the case  $h(n) = \frac{1}{n}$ .

Segal [3] proved the limitation theorem for (I) summability. If  $\sum a_n$  is (I) summable, then  $\sum_{n < x} a_n = o(\log x)$  and has shown in the following theorem that his result is best possible

**THEOREM A** [4] Let  $\epsilon(x)$  be any positive function decreasing to 0 monotonically but arbitrarily slowly as  $x \rightarrow \infty$ . Then there exists a series  $\sum a_n$  which is (I) summable and such that

$$\sum_{n < x} a_n \neq 0(\epsilon(x) \log x) \text{ as } x \rightarrow \infty.$$

Sukla [5] has shown an analogous limitation theorem for  $(D, h(n))$  summability.

**THEOREM B.** If  $\sum a_n$  is  $(D, h(n))$  summable then  $\sum_{n < x} a_n = O(\log x)$  if

(i) 
$$H^*(r) = \sum_{n < r} h^*(n) = O(1)$$

and

(ii) 
$$\sum_{v=1}^n |h^*(v)| = O(\log n).$$

It is remarked in that paper that the condition (ii) cannot be dropped. However if we replace (i) by a slightly stronger condition then we get the result to be true without assuming (ii). In section 4 we show that our revised version of Theorem A is best possible.

**2. MAIN RESULTS**

**THEOREM 1.** If  $\sum a_n$  is  $(D, h(n))$  summable then

$$\sum_{n \leq x} a_n = O\left(\sum_{n < x} |h^*(n)|\right) \tag{2.1}$$

if

$$\sum_{n \leq r} h^*(n) = O((\log r)^{-1-\epsilon}) \text{ as } r \rightarrow \infty. \tag{2.2}$$

We will show that (3.1) is a best possible result

**THEOREM 2.** Let  $\epsilon(x)$  be any positive function decreasing to 0 monotonically but arbitrarily slowly as  $x \rightarrow \infty$ . Then there exists a series  $\sum a_n$  which is  $(D, h(n))$  summable and (3.2) holds and

$$\sum_{1 \leq d \leq \frac{r}{[r/2]+1}} |h^*(d)| / \sum_{n \leq r} |h^*(n)| \tag{2.3}$$

does not tend to zero as  $r \rightarrow \infty$  holds and such that

$$\sum_{n \leq x} a_n \neq o\left(\epsilon(x) \sum_{n \leq x} |h^*(n)|\right) \text{ as } x \rightarrow \infty.$$

**PROOF OF THEOREM 1.** For  $m \geq 0$ , let

$$K(m) = \begin{cases} mD(m) & \text{if } m \geq 1 \\ 0 & \text{if } m = 0 \end{cases}$$

then by (1.1) and (1.2) it follows that

$$K(m) = O(m), \text{ as } n \rightarrow \infty, \text{ and} \tag{2.4}$$

$$\sum_{n \leq r} a_n = \sum_{d \leq r} \frac{K(d)}{d} d \left( \frac{H^*\left(\frac{r}{d}\right)}{d} - \frac{H^*\left(\frac{r}{d+1}\right)}{d+1} \right).$$

By (2.4) it is enough to show that

$$\sum_{d \leq r} d \left| \frac{H^*\left(\frac{r}{d}\right)}{d} - \frac{H^*\left(\frac{r}{d+1}\right)}{d+1} \right| = O\left(\sum_{n \leq r} |h^*(n)|\right). \tag{2.5}$$

The left hand side of (2.5) is maximized by

$$\sum_{d \leq r} \left| H^*\left(\frac{r}{d}\right) - H^*\left(\frac{r}{d+1}\right) \right| + \sum_{d \leq r} \frac{|H^*\left(\frac{r}{d+1}\right)|}{d+1}, \tag{2.6}$$

Now

$$\sum_{d \leq r} \left| H^*\left(\frac{r}{d}\right) - H^*\left(\frac{r}{d+1}\right) \right| = \sum_{d \leq r} \left| \sum_{\frac{r}{d+1} \leq v \leq \frac{r}{d}} h^*(v) \right| \leq \sum_{1 \leq v \leq r} |h^*(v)|$$

and

$$\sum_{d \leq r} \frac{|H^*\left(\frac{r}{d+1}\right)|}{d+1} = O\left(\sum_{d \leq r-2} \left(\log \frac{r}{d+1}\right)^{-1-\epsilon} \frac{1}{d+1}\right) = O(1)$$

since  $H^*(x) = O$  for  $x < 1$

**PROOF OF THEOREM 2.** Define  $b_n$  by

$$b_n = \sum_{d|n} h^* \left( \frac{n}{d} \right) \left( \frac{dD(d) - (d-1)D(d-1)}{d} \right), \text{ where } D(t) \rightarrow 0 \text{ as } t \rightarrow \infty \quad (2.7)$$

then

$$\frac{1}{t} \sum_{n < t} n \sum_{d|n} b_d h \left( \frac{n}{d} \right) = \frac{1}{t} \sum_{n < t} n \sum_{d|n} h \left( \frac{n}{d} \right) \sum_{r|d} h^* \left( \frac{r}{d} \right) \left( \frac{rD(r) - (r-1)D(r-1)}{r} \right) = D(t). \quad (2.8)$$

Since  $D(t) \rightarrow 0$ ,  $\sum b_n$  is  $(D, h(n))$  summable to 0.

$$\begin{aligned} \sum_{n \leq r} b_n &= \sum_{n < r} \sum_{d|n} h^* \left( \frac{n}{d} \right) \left( \frac{dD(d) - (d-1)D(d-1)}{d} \right) \\ &= \sum_{d \leq r} \left( \frac{dD(d) - (d-1)D(d-1)}{d} \right) \sum_{\substack{m \leq \frac{r}{d} \\ d|m}} h^*(m) \\ &= \sum_{d \leq r} D(d) - D(d-1) H^* \left( \frac{r}{d} \right) + \sum_{d \leq r} \frac{D(d-1)}{d} H^* \left( \frac{r}{d} \right) = \sum_1 + \sum_2. \end{aligned}$$

Now

$$\sum_1 = \sum_{d \leq r} D(d) \left[ H^* \left( \frac{r}{d} \right) - H^* \left( \frac{r}{d+1} \right) \right].$$

Since  $H^*(x) = 0$  for  $x < 1$

$$\sum_2 = O \left( \sum_{d \leq r} \frac{1}{d} \left| H^* \left( \frac{r}{d} \right) \right| \right) = O(1) \text{ as } r \rightarrow \infty \text{ by (2.2).}$$

We have now

$$\sum_1 = \sum_{n \leq r} b_n + O(1). \quad (2.9)$$

Suppose the theorem does not hold then

$$\sum_{n \leq r} b_n = o \left( \epsilon(r) \sum_{n \leq r} |h^*(n)| \right).$$

So (2.9) becomes

$$\sum_1 = o \left( \epsilon(r) \sum_{n \leq r} |h^*(n)| \right). \quad (2.10)$$

Since  $D(d) \rightarrow 0$  as  $n \rightarrow \infty$ , let

$$\alpha_{r,d} = \frac{1}{\epsilon(r) \sum_{n \leq r} |h^*(n)|} \left[ H^* \left( \frac{r}{d} \right) - H^* \left( \frac{r}{d+1} \right) \right].$$

It is well known that in order for  $\alpha_{r,d}$  to transform all sequences tending to 0 into sequences tending to 0,

$$\frac{1}{\epsilon(r) \sum_{n \leq r} |h^*(n)|} \sum_{d \leq r} \left| H^* \left( \frac{r}{d} \right) - H^* \left( \frac{r}{d+1} \right) \right| < c$$

must hold for all  $r$  where  $c$  is independent of  $r$

$$\sum_{d \leq r} \left| H^* \left( \frac{r}{d} \right) - H^* \left( \frac{r}{d+1} \right) \right| = \sum_{d \leq r} \left| \sum_{\substack{m < \frac{r}{d+1} \\ d+1 < m \leq \frac{r}{d}}} h^*(m) \right| \geq \sum_{r^{1/2} < d \leq r} \left| \sum_{\substack{m < \frac{r}{d+1} \\ d+1 < m \leq \frac{r}{d}}} h^*(m) \right|.$$

Since in this last sum  $\frac{r}{d} - \frac{r}{d+1} < 1$  the inner sum contains at most one term, and so

$$\frac{1}{\epsilon(r) \sum_{n \leq r} |h^*(n)|} \sum_{d \leq r} \left| H^* \left( \frac{r}{d} \right) - H^* \left( \frac{r}{d+1} \right) \right| \geq \frac{1}{\epsilon(r)} \left( \frac{\sum_{1 < d \leq \frac{r}{r+7\epsilon(r)}} |h^*(d)|}{\sum_{n \leq r} |h^*(n)|} \right)$$

tends to infinity as  $r \rightarrow \infty$  since by (2.3) the expression in the bracket does not tend to zero as  $r \rightarrow \infty$ . This completes the proof of Theorem 2

Agnew [6] showed directly that, for  $r > 0$  the Cesàro and Riesz transforms  $C_r(n), R_r(n)$  respectively of a given series  $\sum a_n$  are equiconvergent i.e  $C_r(n), R_r(n)$  exist for each  $n$  and

$$r \xrightarrow{\infty} \infty (C_r(n) - R_r(n)) = 0.$$

These concepts are applied to arithmetic summation methods (I) and  $(D, h(n))$  for particular values of  $h(n)$  by Jukes [7]. He has found different conditions under which the equiconvergence of  $\frac{6}{\pi^2}(I)$  and  $(D, \frac{\mu^2(n)}{n})$  have been established. The  $(D, \frac{\mu^2(n)}{n})$  and  $\frac{6}{\pi^2}(I)$  transform are given by

$$b_{nk} = \frac{k}{n} \sum_{r \leq \frac{n}{k}} \mu^2(r), \quad C_{nk} = \frac{6}{\pi^2} \frac{k}{n} \left\lfloor \frac{n}{k} \right\rfloor$$

respectively. Let  $M_2 = \lim_n \sup \sum k |\Delta(\frac{b_{nk} - c_{nk}}{k})|$

$$A_2 = \lim_n \sup \left| \sum_{k=\infty}^n \frac{ka_k}{(n+1)} \right|.$$

**THEOREM C** [7] Tauherian constants  $M_2$  do not exist for comparisons of conservative matrices with non-conservative matrices

**THEOREM D** [7] The  $(D, \frac{\mu^2(n)}{n})$  and  $6/\pi^2(I)$  transform are not equiconvergent whenever  $A_2 < \infty$

We have proved (see Kuttner and Sukla [8]) that

**THEOREM E.** The  $(D, h(n))$  is conservative if and only if  $\sum_{n=1}^{\infty} |h(n)| < \infty$ . It is to note that if part of the above theorem was proved earlier by Jukes [9]. See S. L. Segal, *Math. Reviews* 86e 11093 (May 1986, p 1864)

**THEOREM 3.** The  $(D, h(n))$  and (I) are not equiconvergent whenever  $A_2 < \infty$  and  $\sum |h(n)| < \infty$

**PROOF.** By Theorem C since (I) is not conservative and  $(D, h(n))$  is conservative for  $\sum |h(n)| < \infty$  whenever  $A_2 < \infty$ ,  $(D, h(n))$  and (I) are not equiconvergent

From Theorem E also we get that the following theorem of Jukes as corollaries

**COROLLARY 1.** The methods  $(D, \frac{\mu(n)}{n})$  and  $(D, \frac{\lambda(n)}{n})$  are not conservative

**PROOF.** Since  $\sum_{n=1}^{\infty} \frac{\mu(n)}{n}$  and  $\sum_{n=1}^{\infty} \frac{\lambda(n)}{n}$  are not absolutely convergent. So by Theorem 3 the result follows.

**COROLLARY 2.**  $(D, \mu^2(n)/n)$  and  $(D, \lambda(n)/\pi^2(n))$  transforms are not equiconvergent whenever  $A_2 < \infty$ .

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