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ON A PROBLEM OF COMMUTATIVITY OF AUTOMORPHISMS

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ABSTRACT. In this note we provide a partial answer to a problem proposed by M. Brešar. We prove that if α,β are automorphisms of a commutative prime ring of characteristic not equal to 2 satisfying the equation $\alpha+\alpha^{-1}=\beta+\beta^{-1}$, then either $\alpha=\beta$ or $\alpha=\beta^{-1}$. As a consequence α and β commute and in this situation the equation itself ensures the commutativity of α and β.

KEY WORDS AND PHRASES: Prime ring, automorphism.
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The equation

$$\alpha+\alpha^{-1}=\beta+\beta^{-1} \tag{$*$}$$

where α and β are automorphisms of a von Neumann algebra has been extensively studied This equation (in case α and β commute) has played an important role in the study of Tomita-Takesaki theory [1,2,3]. Several conditions have been considered where the equation $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ itself implies the commutativity of α and β and thus making the additional assumption that α and β commute as redundant. For instance, it has been shown in [4] that if M is a commutative semisimple Banach algebra and α,β are automorphisms of M satisfying equation $(*)$, then an application of Gelfand's theory implies that α and β commute. Also, it has been shown in [5] that if α and β are $*$-automorphisms of a C^*-algebra A satisfying equation $(*)$ and if either α or β is inner, then α and β commute. Recently Brešar [6,7] has studied this equation on prime and semiprime rings and has remarkably extended most of the decomposition results of [4,8] on von Neumann algebras about this equation to semiprime and prime rings, using purely algebraic techniques. As an application of Posner's result for (α,β)-derivations, Brešar [6, Corollary 3] has shown the following generalization of a result of Thaheem [4,8].

THEOREM A. *Let R be a prime ring of characteristic not 2. Suppose that automorphisms α, β of R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. If α and β commute then either $\alpha=\beta$ or $\alpha=\beta^{-1}$.*

In [6] Brešar has proposed an open question whether or not the assumption that α and β commute can be removed in Theorem A. In this note we are precisely concerned with this question and provide a partial answer to his problem. We prove that in case R is commutative then the assumption of commutativity of α and β can indeed be removed from Theorem A. We prove the following theorem:

THEOREM B. *Let R be a commutative prime ring of characteristic not 2. Suppose that automorphisms α,β of R satisfy $\alpha+\alpha^{-1}=\beta+\beta^{-1}$. Then either $\alpha=\beta$ or $\alpha=\beta^{-1}$.*

PROOF. It follows from the equation

$$\alpha+\alpha^{-1}=\beta+\beta^{-1} \tag{1}$$

that for any $x \in R$,

$$(\alpha-\beta)(x^2)=(\beta^{-1}-\alpha^{-1})(x^2). \tag{2}$$

Rewriting (2), we obtain $\alpha(x^2)-\beta(x^2)=\beta^{-1}(x^2)-\alpha^{-1}(x^2)$. That is,

$$(\alpha(x))^2-(\beta(x))^2=\left(\beta^{-1}(x)\right)^2-\left(\alpha^{-1}(x)\right)^2. \tag{3}$$

Since R is commutative, then using (1) we may rewrite (3) as

$$(\alpha(x)-\beta(x))(\alpha(x)+\beta(x))=(\alpha(x)-\beta(x))\left(\beta^{-1}(x)+\alpha^{-1}(x)\right)$$

or what is same

$$(\alpha(x)-\beta(x))\left(\alpha(x)+\beta(x)-\beta^{-1}(x)-\alpha^{-1}(x)\right)=0. \tag{4}$$

By equation (1), we may rewrite (4) as

$$(\alpha(x)-\beta(x))\left(\beta(x)+\beta^{-1}(x)-\alpha^{-1}(x)+\beta(x)-\beta^{-1}(x)-\alpha^{-1}(x)\right) \tag{5}$$

or equivalently

$$2(\alpha(x)-\beta(x))\left(\beta(x)-\alpha^{-1}(x)\right)=0. \tag{6}$$

In view of the commutativity of R, equation (6) implies that for any $y \in R$, $2(\alpha(x)-\beta(x))\,y(\beta(x)-\alpha^{-1}(x))=0$. Since R is prime and characteristic of R is not 2, therefore we have $\alpha(x)-\beta(x)=0$ or $\beta(x)-\alpha^{-1}(x)=0$ for any $x \in R$. Thus either $\alpha=\beta$ or $\alpha=\beta^{-1}$. This completes the proof.

It follows from the conclusion of the above theorem that α and β commute. In other words, the equation $\alpha+\alpha^{-1}=\beta+\beta^{-1}$ ensures the commutativity of α and β. It would be interesting to resolve the problem for certain types of noncommutative prime rings.

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