THE RELATIVE DIHEDRAL HOMOLOGY OF INVOLUTIVE ALGEBRAS

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ABSTRACT. Let $f : A \rightarrow B$ be a homomorphism of involutive algebras A, B . The purpose of this paper is to define a free involutive algebra resolution of algebra B over f and use it to define and study the relative dihedral homology.

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1. Introduction. Let A, B be involutive algebras (an involution $*$ is an anti-automorphism of degree zero and order 2) and let $f : A \rightarrow B$ be a homomorphism. Our first aim is to find a free involutive algebra resolution R of algebra B over the homomorphism $f : A \xrightarrow{i} R \xrightarrow{\pi} B$, where i is an inclusion and π is a quasi-isomorphism. The second aim is to define the relative dihedral homology as

$${}^{\epsilon}\mathcal{H}_{\bullet}(A \xrightarrow{f} B) = \mathcal{H}_{\bullet}\left(\frac{R}{(A + [R, R] + \text{Im}(1 - r^{\epsilon}))}\right), \quad (1.1)$$

where $[R, R]$ is the commutant of algebra R , r^{ϵ} is the involution on R , and study its main properties.

First, we recall some definitions and facts from [4, 5]. Let A be an associative algebra over a field k ($k = \mathbb{R}$ or $\mathbb{C}$). Define the complex $C(A) = (C_{\bullet}(A), \mathcal{C}_{\bullet})$, where $C_n(A) = A \otimes \cdots \otimes A$ is the tensor product of algebra A ($n+1$ times) and $\mathcal{C}_n : C_n(A) \rightarrow C_{n-1}(A)$ is the boundary operator

$$\mathcal{C}_n(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}. \quad (1.2)$$

It is well known that $\mathcal{C}_{n-1}\mathcal{C}_n = 0$, that is, the complex $C(A)$ is a chain complex. This complex is called the Hochschild (simplicial) complex and its homology is called the Hochschild homology ($\mathcal{H}_{\bullet}(A)$). If A is a unital involutive algebra, then on the complex $C(A)$, one acts by the operators $t_n, r_n : C_n(A) \rightarrow C_n(A)$ by means of

$$\begin{aligned} t_n(a_0 \otimes \cdots \otimes a_n) &= (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}, \\ r_n(a_0 \otimes \cdots \otimes a_n) &= (-1)^{n(n+1)/2} \epsilon a_0^* \otimes a_n^* \otimes \cdots \otimes a_1^*, \quad \epsilon = \pm 1. \end{aligned} \quad (1.3)$$

Consider the quotient complex $\mathcal{C}\mathcal{D}_n(A) = C_n(A) / \text{Im}(1 - t_n) + \text{Im}(1 - r_n)$ of a complex $C_n(A)$. Following [3] the dihedral homology of algebra, A is the homology of the complex $\mathcal{C}\mathcal{D}_{\bullet}(A)$.

2. Free involutive algebra resolution. In this part, we discuss the existence of the free involutive algebra resolution. Let $E = \sum_{n=0}^{\infty} E_n$ be a graded involutive vector space over a field K . Suppose that R is a differential graded (in short DG) involutive K -algebra and let $R\langle E \rangle = R * T_k(E)$ be the free product of algebras, where $T_k(E) = \sum_{j \geq 0} E^{\otimes j}$ is the tensor algebra over K . We define an involution on algebra $R\langle E \rangle$ to be the unique anti-automorphism on $R\langle E \rangle$ which restricts to the given involution on R and E (this is enough thanks to the universal property of the tensor algebra and the free product). The product in $R\langle E \rangle$ is given by

$$(r_1 e_1 \cdots r_n e_n r_{n+1}) \cdot (\hat{r}_1 \hat{e}_1 \cdots \hat{r}_k \hat{e}_k \hat{r}_{k+1}) = (r_1 e_1 \cdots r_n e_n (r_{n+1} \hat{r}_1) \hat{e}_1 \cdots \hat{r}_k \hat{e}_k \hat{r}_{k+1}),$$

$$r_i, \hat{r}_j \in R, \quad e_i, \hat{e}_j \in T_k(E), \quad (r^*)^* = (e^*)^* = e. \quad (2.1)$$

DEFINITION 2.1. Let $f : R_1 \rightarrow R_2$ be a homomorphism of involutive differential graded K -algebras. An algebra R_2 is a free algebra over the homomorphism f if there exists an isomorphism $\alpha : R_1\langle E \rangle \simeq R_2$, where E is an involutive differential graded vector space with the following commutative diagram:

$$\begin{array}{ccc} R_1 & \xrightarrow{f} & R_2 \\ & \searrow i & \downarrow \wr \alpha \\ & & R_1\langle E \rangle, \end{array} \quad (2.2)$$

where i is the inclusion map.

LEMMA 2.2. Let $f : A \rightarrow B$ be a morphism of involutive K -algebra. Then there exists an involutive differential graded algebra $R = \sum_{i=0}^{\infty} R_i$ with the following properties

(i) π is surjection and the following diagram is commutative

$$\begin{array}{ccc} & & R \\ & \nearrow i & \downarrow \pi \\ A & \xrightarrow{f} & B, \end{array} \quad (2.3)$$

where i is an inclusion map.

There is an amorphism $j : R \rightarrow A$ such that $j \circ i = 1_A$.

(ii) π is quasi-isomorphism, i.e., $\pi_* : \mathcal{H}_\bullet(R) \rightarrow \mathcal{H}_\bullet(B) = B$, where B is a differential graded algebra,

$$(B)_i = \begin{cases} B, & i = 0, \\ 0, & i > 0, \text{ and the differential } \partial^B = 0. \end{cases} \quad (2.4)$$

(iii) The involutive DG algebra R is free over the homomorphism $i : A \rightarrow R$.

DEFINITION 2.3. The involutive DG algebra which satisfies the conditions (i), (ii), and (iii) of Lemma 2.2 is called a free involutive algebra resolution of algebra B over f .

PROOF OF Lemma 2.2.

FIRST STEP. We construct a commutative diagram of involutive algebra

$$\begin{array}{ccc}
 & & R^{(0)} \\
 & \nearrow i_0 & \downarrow \pi_0 \\
 A & \xrightarrow{f} & B,
 \end{array} \quad (2.5)$$

where $R^{(0)}$ is free over the homomorphism $i_0 : A \rightarrow R^{(0)}$, π_0 an involutive surjection. Define $A\langle t_i \rangle = A\langle E(t_i) \rangle$, where $E(t_i)$ is an involutive vector space generated by $\{t_i\}$, or generated by the family $\{t_i, t_i^*\}$. The automorphism $*$: $E(t_i) \rightarrow E(t_i)$ is given as follows: $*(t_i) = (t_i^*)$, $*(t_i^*) = t_i$. We choose a system $\{\mathcal{C}_i^{(0)}\}$ of generators in algebra B . This family is assumed to be closed under an involutive on B .

Now, let $R^{(0)} = A\langle t_i^{(0)} \rangle$, where $t_i^{(0)}$ is equivalent to the generator $\{\mathcal{C}_i^{(0)}\}$ of algebra B , and suppose that $\beta_i^{(0)} = t_i^{(0)}$ or $(t_i^{(0)})^*$. We may define π_0 using the universal property of $R^{(0)}$. Let π_0 be the unique homomorphism of involutive algebras $R^{(0)} \rightarrow B$ which restricts to f on A and sends $t_i^{(0)}$ to $\mathcal{C}_i^{(0)}$.

Since $i_0 : A \rightarrow A\langle t_i^{(0)} \rangle$ is an inclusion map, $i_0(a) = a$, i_0 is an involutive algebra homomorphism and $\pi_0 i_0(a) = \pi_0(a) = f(a)$. Hence, diagram (2.5) is commutative and π_0 is surjective.

Let $j_0 : R^{(0)} \rightarrow A$ be the unique homomorphism involutive algebra restricting to the identity on A and mapping the $t_n^{(0)}$ to zero. $R^{(0)}$ is a DG involutive k -algebra: $(R^{(0)})_0 = R^{(0)}$, $i = 0$, $(R^{(0)})_i = 0$, $i > 0$, $\partial^{R^{(0)}} B_{ij}^{(0)} = 0$. The algebra $R^{(0)}$ is free over the homomorphism $i_0 : A \rightarrow R^{(0)}$ since $R^{(0)} = A\langle t^{(0)} \rangle$.

SECOND STEP. We construct the second commutative diagram

$$\begin{array}{ccc}
 & & R^{(1)} \\
 & \nearrow i_1 & \downarrow \pi_1 \\
 A & \xrightarrow{f} & B,
 \end{array} \quad (2.6)$$

where $R^{(1)}$ is a free algebra over i_1 and π_1 is an involutive surjection. Choose a system $\mathcal{C}_j^{(1)}$ of generators of $\ker \pi_0$ which is closed under the involution. Let $t_j^{(1)}$ be indeterminates which are in bijection with the $\mathcal{C}_j^{(1)}$. Define $R^{(1)} = A\langle t_i^{(0)}, t_j^{(1)} \rangle$, where $t_i^{(0)}$ is defined above. Suppose that $\beta_j^{(1)}$ denotes $t_j^{(1)}$ or $(t_j^{(1)})^*$. The homomorphism π_1 is defined to be the unique homomorphism of involutive algebra $R^{(1)} \rightarrow B$ restricting to π_0 on $R^{(0)}$ and sending $t_j^{(1)}$ to 0. We can see, from the above discussion, that the homomorphism π_1 can be defined as π_0 and that π_1 is surjective since $\pi_1(\beta_1^{(0)}) = \mathcal{C}_1$, $\pi_1(\beta_j^{(1)}) = 0$. The homomorphism $i_1 : A \rightarrow A\langle t_i^{(0)}, t_j^{(1)} \rangle$ is inclusion. The diagram (2.6) is commutative since $(\pi_1 i_1)(a) = \pi_1(a) = f(a)$. The homomorphism j_1 is defined to be unique homomorphisms: $R^{(1)} \rightarrow A$, of involutive algebras restricting to identity on A and mapping $t_i^{(1)}$ to zero. The algebra $R^{(1)} = A\langle t_i^{(0)}, t_j^{(1)} \rangle$ is free over i_1 . Finally, we have a differential graded algebra

$$R^{(1)} = (R^{(1)})_0 \oplus (R^{(1)})_1 \oplus \cdots, \quad \deg \beta_i^{(1)} = 0, \quad \deg \beta_j^{(1)} = 1. \quad (2.7)$$

Note that the algebra $R^{(1)}$ also has a universal property with respect to derivations (not only homomorphism). This property should be used to define the differential. The differential $\partial^{R^{(1)}}$ of $R^{(1)}$ is the unique derivation on $R^{(1)}$ satisfying the graded Leibniz rule and commuting with the involution which restricts to zero on $R^{(1)}$ and sends $t_j^{(1)}$ to $\mathcal{C}_j^{(1)}$. So, $\partial^{R^{(1)}} \beta_i^{(0)} = 0$, $\partial^{R^{(1)}} \beta_i^{(0)} = \mathcal{C}_j^{(1)} \in \ker \pi_0$, $\partial_i^{R^{(1)}} = 0$, $i > 1$.

In the same manner, we can construct the commutative diagram

$$\begin{array}{ccc} & & R^{(2)} \\ & \nearrow i_2 & \downarrow \pi_2 \\ A & \xrightarrow{f} & B, \end{array} \quad (2.8)$$

where $R^{(2)} = A\langle t_i^{(0)}, t_j^{(1)}, t_k^{(2)} \rangle$ is a DG algebra, free over i_2 , $R^{(2)} = (R^{(2)})_0 \oplus (R^{(2)})_1 \oplus (R^{(2)})_2 \oplus \dots$, $\deg \beta_i^{(0)} = 0$, $\deg \beta_j^{(1)} = 1$, $\deg \beta_k^{(2)} = 2$, the differential of algebra $R^{(2)}$ is also defined by using a universal property and, hence, $\partial_0^{R^{(2)}} \beta_i^{(0)} = 0$, $\partial_1^{R^{(2)}} \beta_j^{(0)} = \mathcal{C}_j^{(1)}$, $\partial_1^{R^{(2)}} \beta_j^{(2)} = \mathcal{C}_k^{(2)}$, $\partial_i^{R^{(2)}} = 0$, $i > 2$.

Consequently, we can construct an involutive algebra $R^{(i)}$, $i \geq 0$ with the following commutative diagram:

$$\begin{array}{ccccccc} & & R^{(0)} & \xrightarrow{P_0} & R^{(1)} & \xrightarrow{P_1} & \dots \xrightarrow{P_{n-1}} R^{(n)} \xrightarrow{P_0} \dots \\ & \nearrow i_0 & \downarrow \pi_0 & & \downarrow \pi_i & & \downarrow \pi_n \\ A & \longrightarrow & B & \xlongequal{\quad} & B & \xlongequal{\quad} & \dots \xlongequal{\quad} B \xlongequal{\quad} \dots \end{array} \quad (2.9)$$

where π_i is an involutive surjection, $i \geq 0$, $i_n = P_{n-1} \circ \dots \circ P_0 \circ i_0$ is an inclusion map from A to $R^{(n)}$, P_i is also an inclusion map from

$$P_i : A\langle t_{m_0}^{(0)}, t_{m_1}^{(1)}, \dots, t_{m_i}^{(i)} \rangle \quad \text{to} \quad A\langle t_{m_0}^{(0)}, t_{m_1}^{(1)}, \dots, t_{m_i}^{(i)}, t_{m_{i+1}}^{(i+1)} \rangle. \quad (2.10)$$

Define $i_n = q_n \circ \dots \circ q_i \circ j_0$, where q_n is the projection of the map from $A\langle t_{m_0}^{(0)}, t_{m_1}^{(1)}, \dots, t_{m_i}^{(i)} \rangle$ on $A\langle t_{m_0}^{(0)}, t_{m_1}^{(1)}, \dots, t_{m_i}^{(i)}, t_{m_{i+1}}^{(i+1)} \rangle$. The diagram (2.9) is commutative since $i_{n+1}(\beta_i^{(n)}) = \pi_n(\beta_i^{(n)}) = 0$, $n \geq 0$. Define $R = \lim R_n$, $\pi = \lim \pi_n$, $i = \lim i_n$, $j = \lim j_n$. Then the DG algebra R satisfies the items of Lemma 2.2 since

(1) $\pi = \lim i_n$ is an involutive surjection, the diagram

$$\begin{array}{ccc} & & R^{(2)} \\ & \nearrow i_2 & \downarrow \pi_2 \\ A & \xrightarrow{f} & B \end{array} \quad (2.11)$$

is commutative since $i(a) = a$, $\pi(a) = f(a)$.

(2) π is a quasi-isomorphism of DG algebras

$$\begin{array}{ccccccc} (R)_0 & \xleftarrow{\partial_0^R} & (R)_1 & \xleftarrow{\partial_1^R} & \dots & \xleftarrow{\partial_n^R} & (R)_n \xleftarrow{\partial_{n+1}^R} \dots \\ \downarrow \pi_0 & & \downarrow \pi_1 & & & & \downarrow \pi_n \\ B & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & \dots & \xleftarrow{\quad} & 0 \xleftarrow{\quad} \dots \end{array} \quad (2.12)$$

where $\partial_i^R = \lim \partial_i^R$, $(R)_0 = \ker(\pi)_0 = B$, $\text{Im } \partial^R = \ker \partial_0^R$, i.e., $\mathcal{H}_0(R) = B$, $\mathcal{H}_i(R) = 0$.

(3) The DG involutive algebra R is free over the homomorphism $i : A \rightarrow R$ since $R = \langle E \rangle$, E is an involutive vector space generated by the system

$$\{t_{i_0}^{(0)}, t_{i_1}^{(1)}, \dots, t_{i_n}^{(n)}, \dots\}. \quad (2.13)$$

□

3. The relative dihedral homology. In this part, we define the relative dihedral homology and study its properties. Let f be a morphism of involutive algebras A and B over a field K with characteristic zero. Let R_f^B be a free involutive algebra resolution of algebra B over f and, for $r_1, r_2 \in R_f^B$, let $[r_1, r_2] = r_1 r_2 - (-1)^{|r_1||r_2|} r_2 r_1$, where $|r_i| = \deg r_i$, $i = 1, 2$. Let $\mathcal{C} = [R_f^B, R_f^B]$ be the linear space generated by $[r_1, r_2]$, $r_1, r_2 \in R_f^B$. It is clear that, $\mathcal{C} = [R_f^B, R_f^B]$ is a K -submodule of a K -module R_f^B . We construct the complex $([R_f^B, R_f^B] + \text{Im}(1 - r^\epsilon))$, where $r^\epsilon(P) = \epsilon(-1)^{|P|(|P|-1)/2} P^*$, $*$ is an involutive on R_f^B , $\epsilon = \pm 1$. It is clear, from the definition of R_f^B , that $\text{Im}(1 - r^\epsilon)$ is a subcomplex of R_f^B . We have

$$\begin{aligned} \partial[r_1 r_2] &= r_1 r_2 - (-1)^{|r_1||r_2|} r_2 r_1 \\ &= \partial r_1 r_2 + (-1)^{|r_1|} r_1 \partial r_2 - (-1)^{|r_1||r_2|} (\partial r_2 r_1 + (-1)^{|r_2|} r_2 \partial r_1) \\ &= \partial r_1 r_2 - (-1)^{|r_2|(|r_1|+1)} r_2 \partial r_1 + (-1)^{|r_1|} (r_1 \partial r_2 - (-1)^{|r_1|(|r_2|+1)} \partial r_2 r_1) \\ &= [\partial r_1, r_2] + (-1)^{|r_1|} [r_1, \partial r_2], |\partial r_i| = |r_i| - 1, \quad i = 1, 2. \end{aligned} \quad (3.1)$$

Then $[R_f^B, R_f^B]$ is a subcomplex in R_f^B . Therefore, the chain complex of the K -module $[R_f^B, R_f^B] + \text{Im}(1 - r^\epsilon)$ is a subcomplex of R_f^B .

DEFINITION 3.1. Let $f : A \rightarrow B$ be an involutive K -algebra ($\text{char } K = 0$) homomorphism, R_f^B be a free involutive algebra resolution of algebra B over f . Then the relative dihedral homology is defined as follows:

$${}^\epsilon \mathcal{H}\mathcal{D}_i(A \xrightarrow{f} B) = \mathcal{H}_i \left(\frac{R_f^B}{[R_f^B, R_f^B] + \text{Im}(1 - r^\epsilon)} \right). \quad (3.2)$$

The main properties of the relative dihedral homology are submitted in Theorems 3.2, 3.6, and 3.7.

THEOREM 3.2. Let A be an involutive algebra. Then ${}^\epsilon \mathcal{H}\mathcal{D}_i(A \rightarrow 0) = {}^\epsilon \mathcal{H}\mathcal{D}_{i-1}(A)$, where ${}^\epsilon \mathcal{H}\mathcal{D}_i(A)$ is the dihedral homology of k -algebra A ($\text{char}(k) = 0$).

PROOF. To do this, we need the following definition and lemmas.

DEFINITION 3.3. The K -algebra $A\langle t \rangle$, generated by the elements $a_0 t a_1 t \cdots t a_n$, $n \geq 0$, can be considered as an involutive DG algebra by requiring that the morphism $A \rightarrow A\langle t \rangle$ is a morphism of involutive differential graded algebras (A is viewed as a DG algebra concentrated in degree 0) and that $\deg t = 1$, $\partial t = 0$, and $t^* = t$.

LEMMA 3.4. The algebra $A\langle t \rangle$ is splitable and is a free involutive algebra resolution of the algebra $B = 0$ over the homomorphism $A \rightarrow 0$.

PROOF. Define the following chain complex

$$A \xleftarrow{\partial} AtA \xleftarrow{\partial} AtAtA \xleftarrow{\partial} \dots \xleftarrow{\partial} At \dots tA \xleftarrow{\partial} \dots, \quad (3.3)$$

where $At \dots tA$ (t - n -times) is a K -module and the boundary operator ∂ is given by

$$\begin{aligned} \partial(a_0ta_1t \dots ta_{n-1}ta_n) &= \sum_{i=0}^{n-1} (-1)^i a_0ta_1t \dots ta_i(\partial t)a_{i+1}t \dots ta_n \\ &= \sum_{i=0}^{n-1} (-1)^i a_0ta_1t \dots t(a_ia_{i+1})t \dots ta_n. \end{aligned} \quad (3.4)$$

Note that the differential ∂ in $a\langle t \rangle$ is equivalent to the operator $\mathcal{C}'_n : C_n(A) \rightarrow C_{n-1}(A)$ (see [4]), defined by

$$\mathcal{C}'_n(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_ia_{i+1} \otimes \dots \otimes a_n. \quad (3.5)$$

Following [4], the complex $(C_n(A), \mathcal{C}'_n)$ is splitable and so the complex $A\langle t \rangle$ is also splitable, that is, $\mathcal{H}_*(A\langle t \rangle) = 0$. Therefore, the algebra $A\langle t \rangle$ is a free involutive algebra resolution of the algebra $B = 0$ over the homomorphism $A \rightarrow 0$. $\square$

LEMMA 3.5. *The complex $A\langle t \rangle/[A, A\langle t \rangle]$ is the standard simplicial (Hochschild) complex.*

PROOF. Consider the factor complex $A\langle t \rangle/[A, A\langle t \rangle]$. The complex $A\langle t \rangle/[A, A\langle t \rangle]$ is generated by the elements $a_0ta_1t \dots a_{n-1}t$, since $a_0ta_1t \dots ta_{n-1}ta_n = a_na_0 \times ta_1t \dots ta_{n-1}t \pmod{[A, A\langle t \rangle]}$. The action of the differential ∂ on the complex $A\langle t \rangle/[A, A\langle t \rangle]$ is given by

$$\begin{aligned} \partial(a_0ta_1t \dots ta_{n-1}ta_n) &= \sum_{i=0}^{n-1} (-1)^i a_0ta_1t \dots t(a_ia_{i+1})t \dots a_{n-1}ta_nt \\ &\quad + (-1)^n a_0ta_1t \dots a_{n-1}ta_n \\ &= \sum_{i=0}^{n-1} (-1)^i a_0ta_1t \dots t(a_ia_{i+1})t \dots a_{n-1}ta_nt \\ &\quad + (-1)^n a_na_0ta_1t \dots a_{n-1}t. \end{aligned} \quad (3.6)$$

Consider the complex

$$A \xleftarrow{\text{id}} A \xleftarrow{\mathcal{C}} A^{\otimes 2} \xleftarrow{\mathcal{C}} \dots \xleftarrow{\mathcal{C}} A^{\otimes n} \xleftarrow{\mathcal{C}} \dots, \quad (3.7)$$

where $\mathcal{C}$ is the differential in the standard Hochschild complex (see [4]). Since the space $(A\langle t \rangle/[A, A\langle t \rangle])_{n+1}$ identifies with the space

$$A^{\otimes n+1} : a_0ta_1 \dots ta_nt \rightarrow a_0 \otimes a_1 \otimes \dots \otimes a_n, \quad (3.8)$$

and the differential in $A\langle t \rangle/[A, A\langle t \rangle]$ identifies with the differential in the standard Hochschild complex, $A\langle t \rangle/A + [A, A\langle t \rangle]$ is the Hochschild (simplicial) complex of algebra A . $\square$

Now, we prove Theorem 3.2. Consider the factor complex: $A\langle t \rangle / [A\langle t \rangle, A\langle t \rangle] + \text{Im}(1 - r^\epsilon)$, such that

$$\begin{aligned} a_0 t a_1 t \cdots t a_{n-1} t &= (-1)^n a_n t a_0 t a_1 t \cdots a_{n-1} t, \\ a_0 t a_1 t \cdots t a_{n-1} t &= (-1)^{n(n+1)/2} \epsilon t a_n^* t a_{n-1}^* \cdots t a_1^* t a_0^* \\ &= (-1)^{n(n+1)/2} \epsilon t a_0^* t a_n^* t \cdots t a_1^* t, \end{aligned} \quad (3.9)$$

where $\epsilon = \pm 1$, $\deg a_0 t a_1 t \cdots t a_{n-1} t = n$, $\deg(a_n t) = 1$, $\deg(a_n^*) = 0$, $\deg a_0 t \cdots a_n t = n + 1$. The dihedral homology of $\langle t \rangle$ is the homology of the factor complex $A\langle t \rangle / [A\langle t \rangle, A\langle t \rangle] + \text{Im}(1 - r^\epsilon)$. By factoring $A\langle t \rangle$, first by the subcomplex $A \leftarrow 0 \leftarrow 0 \leftarrow \cdots$ and then by the subcomplex $[A\langle t \rangle, A\langle t \rangle] + \text{Im}(1 - r^\epsilon)$, we get a homomorphism ${}^\epsilon C\mathcal{D}_\bullet(A \rightarrow 0) \rightarrow {}^\epsilon C\mathcal{D}_{\bullet-1}(A)$, which induces an isomorphism in the dihedral homology groups ${}^\epsilon \mathcal{H}\mathcal{D}_i(A \rightarrow 0) \rightarrow {}^\epsilon \mathcal{H}\mathcal{D}_{-1}(A)$. $\square$

THEOREM 3.6. *Let $f : A \rightarrow B$ be a homomorphism of involutive algebras over a field K ($\text{char } K = 0$). Then the relative dihedral homology ${}^\epsilon \mathcal{H}\mathcal{D}_i(A \xrightarrow{f} B)$ does not depend on the choice of the resolution.*

PROOF. The homomorphism f induces a homomorphism of chain complexes

$$f_\bullet : {}^\epsilon C\mathcal{D}_\bullet(A) \longrightarrow {}^\epsilon C\mathcal{D}_\bullet(B), \quad (3.10)$$

where ${}^\epsilon C\mathcal{D}_\bullet(A)$ is the dihedral complex. Consider the diagram

$$\begin{array}{ccc} & & R_f^B \\ & \nearrow i & \downarrow \pi \\ A & \xrightarrow{f} & B, \end{array} \quad (3.11)$$

where R_f^B is defined above, i is the inclusion map. The idea of the proof is to show that the cone of the map i is quasi-isomorphic to an arbitrary category (see [2]), to the complex: $R_f^B / [R_f^B, R_f^B] + \text{Im}(1 - r^\epsilon)$. Since

$$\mathcal{H}_i(R_f^B) = \begin{cases} B, & i = 0, \\ 0, & i > 0, \end{cases} \quad (3.12)$$

then the isomorphism $\pi_\bullet : {}^\epsilon C\mathcal{D}_\bullet(R_f^B) \rightarrow {}^\epsilon C\mathcal{D}_\bullet(B)$ induces an isomorphism of the homology of these complexes. Since $i_\bullet : {}^\epsilon C\mathcal{D}_\bullet(A) \rightarrow {}^\epsilon C\mathcal{D}_\bullet(R_f^B)$ is an inclusion, then $M(i_\bullet) \approx {}^\epsilon C\mathcal{D}_\bullet(R_f^B) / {}^\epsilon C\mathcal{D}_\bullet(A)$, where $M(i_\bullet)$ is the cone of the map i (see [6]).

Note that the symbol $\approx$ always denotes a quasi-isomorphism. It is clear, from the above discussion, that the following diagram is commutative

$$\begin{array}{ccc} & & {}^\epsilon C\mathcal{D}_\bullet(R_f^B) \\ & \nearrow i_\bullet & \downarrow \pi_\bullet \\ {}^\epsilon C\mathcal{D}_\bullet(A) & \xrightarrow{f_\bullet} & {}^\epsilon C\mathcal{D}_\bullet(B), \end{array} \quad (3.13)$$

and, hence, $M(f_\bullet) \approx {}^\epsilon C\mathcal{D}_\bullet(R_f^B)/{}^\epsilon C\mathcal{D}_\bullet(A)$. Following [1], we have $CC_\bullet(R_f^B)/CC_\bullet(A) \approx R_f^B/A + [R_f^B, R_f^B]$ and by using the spectral sequence $E_{ij}^2 = {}^\epsilon \mathcal{H}_\bullet(\mathbb{Z}/2, \mathcal{H}C_j(R_f^B)) = {}^\epsilon \mathcal{H}D_{i+j}(R_f^B)$, we have

$$\frac{{}^\epsilon C\mathcal{D}_\bullet(R_f^B)}{{}^\epsilon C\mathcal{D}_\bullet(A)} \approx \frac{R_f^B}{A + [R_f^B, R_f^B] + \text{Im}(1 - r^\epsilon)}. \quad (3.14)$$

So, $M(f_\bullet) \approx R_f^B/A + [R_f^B, R_f^B] + \text{Im}(1 - r^\epsilon)$. Then ${}^\epsilon \mathcal{H}\mathcal{D}_i(A \xrightarrow{f} B)$ does not depend on the choice of R_f^B . $\square$

THEOREM 3.7. *Let A, B , and C be involutive algebras. Then the following sequence $A \xrightarrow{f} B \xrightarrow{g} C$ induces the long exact sequence of the relative dihedral homology*

$$\cdots \rightarrow {}^\epsilon \mathcal{H}\mathcal{D}_i(A \xrightarrow{f} B) \rightarrow {}^\epsilon \mathcal{H}\mathcal{D}_i(A \xrightarrow{g \circ f} C) \rightarrow {}^\epsilon \mathcal{H}\mathcal{D}_i(B \xrightarrow{g} C) \rightarrow {}^\epsilon \mathcal{H}\mathcal{D}_{i-1}(A \xrightarrow{f} B) \rightarrow \cdots \quad (3.15)$$

PROOF. In Theorem 3.6, it has been proved that any homomorphism $f : A \rightarrow B$ of involutive algebras in an arbitrary category is equivalent to an inclusion $i : A \rightarrow R_f^B$

$$\begin{array}{ccc} & & R_f^B \\ & \nearrow i & \downarrow \wr \\ A & \xrightarrow{f} & B. \end{array} \quad (3.16)$$

Then, for an arbitrary sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of involutive algebras, we have the following complex

$$\begin{array}{ccccc} A & \xrightarrow{i} & R_f^B & \xlongequal{\quad} & B & \xrightarrow{i'} & R_g^C \\ & \searrow f & \parallel & & \searrow g & \parallel & \\ & & B & & & C. \end{array} \quad (3.17)$$

Consider the following sequence of mapping cones

$$0 \rightarrow M(i_\bullet) \rightarrow M(i'_\bullet) \rightarrow M(i'_\bullet 0 i_\bullet) \rightarrow 0. \quad (3.18)$$

In general, the sequence (3.18) is not exact. In fact, the composition of two morphisms will be non zero. However, the cone over the morphism $M(i_\bullet) \rightarrow M(i'_\bullet)$ is canonically homotopy equivalent to $M(i'_\bullet 0 i_\bullet)$. So, we get the following long exact sequence of the relative dihedral homology

$$\begin{aligned} \cdots \rightarrow {}^\epsilon \mathcal{H}\mathcal{D}_i(A \xrightarrow{f} B) &\rightarrow {}^\epsilon \mathcal{H}\mathcal{D}_i(A \xrightarrow{g \circ f} C) \\ &\rightarrow {}^\epsilon \mathcal{H}\mathcal{D}_i(B \xrightarrow{g} C) \rightarrow {}^\epsilon \mathcal{H}\mathcal{D}_{i-1}(A \xrightarrow{f} B) \rightarrow \cdots \end{aligned} \quad (3.19) \quad \square$$

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