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WEBBED SPACES, DOUBLE SEQUENCES, AND THE MACKEY CONVERGENCE CONDITION

ARMANDO GARCÍA-MARTÍNEZ

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ABSTRACT. In [3], Gilsdorf proved, for locally convex spaces, that every sequentially webbed space satisfies the Mackey convergence condition. In the more general frame of topological vector spaces, this theorem and its inverse are studied. The techniques used are double sequences and the localization theorem for webbed spaces.

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1. Introduction. A *web* W in a topological vector space E is a countable family of balanced subsets of E, arranged in *layers*. The first layer of the web consists of a sequence $(A_p : p = 1,2,\dots)$ whose union absorbs each point of E. For each set A_p of the first layer, there is a sequence $(A_{pq} : q = 1,2,\dots)$ of sets, called the sequence determined by A_p, such that

$$A_{pq} + A_{pq} \subset A_p \quad \text{for each } q; \tag{1}$$

$$\bigcup \{A_{pq} : q = 1,2,\dots\} \text{ absorbs each point of } A_p. \tag{2}$$

Further, layers are made up in a corresponding way such that each set of the kth layer is indexed by a finite row of k integers and, at each step, the above mentioned two conditions are satisfied. Suppose that one chooses a set A_p from the first layer, then a set A_{pq} of the sequence determined by A_p and so on. The resulting sequence $S = (A_p, A_{pq}, A_{pqr},\dots)$ is called a strand. Whenever we are dealing with only one strand, we can simplify the notation by writing $W_1 = A_p$, $W_2 = A_{pq}$, etc. Thus, $S = (W_k)$ is a strand, where, for each k, W_k is a set of the kth layer.

Let $S = (W_k)$ be a strand. Consider $x_k \in W_k$ and the series $\sum_{k=1}^{\infty} x_k$. The space E is *webbed* if the series $\sum_{k=1}^{\infty} x_k$ is convergent for any choice of $x_k \in W_k$; and E is *strictly webbed* if $\sum_{k=n+1}^{\infty} x_k$ converges to some $x \in W_n$ for every $n \in \mathbb{N}$ and for any choice of $x_k \in W_k$. The standard references for webs in a topological vector space are [5, 7, 8].

Let (E,τ) be a topological vector space. $(x_n)_n \subset E$ is a *Mackey null sequence* if there exists a sequence of real numbers $(r_n)_n$ such that $r_n \to \infty$ and $r_n x_n \to 0$ in E. We say that $(x_n)_n \subset E$ is *Mackey convergent* to x if $(x_n - x)_n$ is a Mackey null sequence. A topological vector space E satisfies the *Mackey convergence condition (M.c.c.)* if every null sequence is Mackey null.

2. Double sequences. A *completing double sequence* in a topological vector space (E,τ) is a family $(K_j^n)_{n,j\in\mathbb{N}}$ of balanced subsets such that

(1) $K_j^n \subset K_j^{n+1}$ for every n, j natural numbers;

(2) $K_{j+1}^n + K_{j+1}^n \subset K_j^n$ for every n, j natural numbers;

(3) $\bigcup_{n\in\mathbb{N}} K_j^n$ is absorbent in E for every j natural number;

(4) for every $j_0 \in \mathbb{N}$, if $x_j \in K_j^n$ with $j > j_0$, then $\sum_{j=j_0+1}^{\infty} x_j$ converges in E to some $x \in K_{j_0}^n$.

Moreover, $(K_j^n)_{j,n\in\mathbb{N}}$ is *compatible with the topology* if, for each zero neighborhood U in E and for every natural number n, there exists a natural number J such that $K_j^n \subset U$ for every $j \geq J$.

For example, if E is sequentially complete and has a fundamental sequence of closed bounded sets $A_1 \subset A_2 \subset \cdots$ such that, for each bounded set $B \subset E$, there exists $n_0 \in \mathbb{N}$ such that $B \subset A_{n_0}$ (this is the case if E is the strong dual of a metrizable space). In this case, we define $K_j^n = 2^{-j} A_n$ and it is easy to verify the properties (1) to (4), above. The reader can find further information concerning double sequences in [6].

A topological vector space (E,τ), with a compatible completing double sequence (K_j^n), has a *Sequential Double Sequence* or the *SDS* property if, for each $x_m \to 0$ in E, there exists $n_0 \in \mathbb{N}$ such that, for each j, there exists a natural number M_j such that $x_m \in K_j^{n_0}$, for every $m \geq M_j$.

Theorem 1. *Let (E,τ) be a topological vector space with the SDS property. Then E satisfies the Mackey convergence condition.*

Proof. Let $x_m \to 0$ in (E,τ). Let (K_j^n) be a sequential double sequence, then there exists $n_0 \in \mathbb{N}$ such that, for every j, there exists a natural number M_j such that $x_m \in K_j^{n_0}$, for every $m \geq M_j$. For $n, j \in \mathbb{N}$, we have $K_{j+1}^n \subset (1/2)K_j^n$; so $K_{j+2}^n \subset (1/2)K_{j+1}^n \subset (1/2^2)K_j^n$. Consequently, for each $l \in \mathbb{N}$, $K_{j+1}^n \subset (1/2^l)K_j^n$. Note that $(1/2^j) \leq (1/j)$, for every $j \in \mathbb{N}$ and $K_{j+j}^{n_0} = K_{2j}^{n_0} \subset (1/2^j)K_j^{n_0} \subset (1/j)K_j^{n_0}$. So, there exists $M_{2j} \in \mathbb{N}$ such that $x_m \in K_{2j}^{n_0} \subset (1/2^j)K_j^{n_0} \subset (1/j)K_j^{n_0}$, for every $m \geq M_{2j}$; which implies that $jx_m \in K_j^{n_0}$, for every $m \geq M_{2j}$. Analogously, for $(j+1)$, there exists $M_{2(j+1)} \geq M_{2j}$ such that $(j+1)x_m \in K_{j+1}^{n_0}$, for every $m \geq M_{2(j+1)}$; and so, for all $j \in \mathbb{N}$. Define $r_m = j$ if $M_{2j} \leq m < M_{2(j+1)}$, then $\lim_{m\to\infty} r_m = \lim_{j\to\infty} j = \infty$. Since (K_j^n) is compatible with the topology, we conclude that $r_m x_m \to 0$. □

From the theorem, a space with the SDS property is a space with the Mackey convergence condition. In what follows, we study the conditions under which we have an equivalence of these two properties. First, let us introduce another type of double sequences: a topological vector space (E,τ), with a compatible completing double sequence (K_j^n), has a *quasi-Sequential Double Sequence* or the qSDS property if, for each $x_n \to 0$ in E, there exists n_0 such that, for every j, there exists a natural number M_j and a positive real number α_j such that $m > M_j$ implies that $x_m \in \alpha_j K_j^{n_0}$.

If $\alpha_j = 1$, for every j, in a qSDS, then it becomes on SDS. So, the qSDS is more general than the SDS. The next proposition gives the condition for the equivalence.

Proposition 2. *Let (E,τ) be a topological vector space with the Mackey convergence condition. Then the SDS and the qSDS are the same.*

Proof. Let $x_m \to 0$ in a space (E,τ) with qSDS property. By the Mackey convergence condition, there exists a scalar sequence $r_m \to \infty$ such that $r_m x_m \to 0$. Then there exists n_0 such that $r_m x_m \in \alpha_j K_j^{n_0}$, for some $\alpha_j > 0$ whenever $m \geq M_j$. Hence, $x_m \in (\alpha_j/r_m)K_j^{n_0} \subset K_j^{n_0}$ if $m \geq M_j$ and $r_m \geq \alpha_j$. □

Next, we see an example, where the qSDS property holds and the SDS property does not.

Let $(E, \|\cdot\|)$ be a Banach space with a sequence $(x_m)_{m\in\mathbb{N}}$ weakly convergent to zero and not norm convergent. Let B be the closed unit ball in E. For each $n, j \in \mathbb{N}$, let $K_j^n = 2^{-j}B$. Then (K_j^n) is a compatible completing double sequence with respect to the norm topology and, consequently, with respect to any weaker topology τ, especially the weak topology since the map $i : (E, \|\cdot\|) \to (E,\tau)$ is continuous. Now, $(x_m)_{m\in\mathbb{N}}$ is not contained in K_j^n, since K_j^n are neighborhoods in the norm topology such that $\bigcap_j K_j^n = \{0\}$ and, by [4, Ex. 4] and [4, cor. of Thm. 3], (E,σ) does not have the M.c.c. Nevertheless, $(x_m)_{m\in\mathbb{N}}$ is bounded with respect to both the weak and norm topologies. So, for every K_j^n, there exists α_j such that $(x_m)_m \subset \alpha_j K_j^n$.

We have the following implication: SDS ⟹ qSDS. This implication can be reversed if the space has the M.c.c. Furthermore SDS ⟹ M.c.c. So, we have the following corollary:

Corollary 3. *Let E be a topological vector space with a compatible completing double sequence. Then E has SDS property if and only if the qSDS property and M.c.c. hold.*

3. Mackey convergence and sequentially webbed spaces.

E is *sequentially webbed* if it has a compatible web W such that, for every null sequence $(x_n)_{n\in\mathbb{N}}$ in E, there exists a finite collection of strands $\{(W_k^{(1)}),\ldots,(W_k^{(m)})\}$ of W such that, for every natural number k, there exists M_k such that $n \geq M_k$ implies $x_n \in \bigcup_{i=1}^m W_k^{(i)}$. Gilsdorf [3] proved two relations between the M.c.c. and the sequentially webbed spaces in the locally convex case.

Here, we generalize these results. One to topological vector spaces and the other to locally r-convex spaces. In fact, the concept of webbed spaces, introduced here, does not use local convexity. Note that in this case, in each strand, we have $2W_{k+1} \subset W_{k+1} + W_{k+1} \subset W_k$ so that $W_{k+1} \subset 2^{-1}W_k$, and then following the proof of [3, Thm. 12], we have: *if (E,τ) is a sequentially webbed topological vector space, then E has the M.c.c.*

In order to obtain a converse of this result, we need to use a localization theorem [5, Thm. 5.6.3.].

Let $0 < r \leq 1$ fixed. $A \subset E$ is *r-convex* if $\lambda A + \mu A \subset A$, for every $\lambda, \mu \geq 0$ such that $\lambda^r + \mu^r = 1$. Moreover, if A is balanced, we say that A is *absolutely r-convex*. If $r = 1$, we have the usual convexity definition.

For $U \subset E$ balanced and absorbent, let $q_u : E \to \mathbb{R}^+$ be the Minkowski functional defined by $x \to \inf\{\rho > 0 : x \in \rho U\}$. q_u is an *r-seminorm* if $q_u(x+y)^r \leq q_u(x)^r + q_u(y)^r$. Furthermore, if $q_u^{-1}(0) = 0$, it is called an *r-norm*. (E,τ) is *locally r-convex* if it has a fundamental system of zero neighborhoods formed by r-convex sets.

Now, we can use the E_B spaces for locally r-convex spaces. (E,τ) locally r-convex space is *locally r-Baire* if, for every bounded set $A \subset E$, there exists B absolutely r-convex and bounded such that $A \subset B$ and the space (E_B, ρ_B) is a Baire space, where E_B

is the span of B and ρ_B is the topology generated by the r-norm q_B^r.

Theorem 4. *Let (E,τ) be a locally r-Baire locally r-convex space and strictly webbed. If E satisfies the Mackey convergence condition, then E is sequentially webbed.*

Proof. Let W be a strict web in E; $(x_n)_n \subset E$ a null sequence, and $r_n \to \infty$ a sequence of real numbers such that $r_n x_n \to 0$ in E. Let $A = \{r_n x_n : n \in \mathbb{N}\}$, A is bounded, then there exists a bounded absolutely r-convex set B such that (E_B, ρ_B) is a Baire space and A is a bounded set in E_B. The identity map $i : E_B \to E$ is continuous. Hence, by the localization theorem, i has a closed graph and there exists a strand (W_k) such that $i^{-1}(W_k) = E_B \cap W_k$ is a zero neighborhood in (E_B, ρ_B) for every k. Finally, $A \subset \alpha_k(E_B \cap W_k) \subset \alpha_k W_k$ for some α_k, a positive real number. So, $r_n x_n \in \alpha_k W_k$ and $x_n \in (\alpha_k / r_n) W_k \subset W_k$, for n sufficiently large such that $|(\alpha_k / r_n)| \leq 1$. □

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García-Martínez: Instituto de Matemáticas, U.N.A.M., Area de la Investigación Científica, Circuito Exterior, Ciudad Universitaria, México, D.F. 04510

E-mail address: agarcia@gauss.math.unam.mx


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