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WEAK CONVERGENCE THEOREM FOR PASSTY TYPE ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we prove a convergence theorem for Passty type asymptotically nonexpansive mappings in a uniformly convex Banach space with Fréchet-differentiable norm.

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1. Introduction. In 1972, Goebel and Kirk [3] introduced the class of asymptotically nonexpansive mappings and proved that every asymptotically nonexpansive self-mapping of a nonempty closed, bounded, and convex subset of a uniformly convex Banach space has a fixed point. After the existence theorem of Goebel and Kirk [3] several authors ([4, 8]) have shown interest in iterative construction of a fixed point of asymptotically nonexpansive mappings in uniformly convex Banach space. In these papers, Opial's condition [5] was a common tool for such construction.

Now, if we consider a space of type L_p, $p \neq 2$, then we find that Opial's condition fails to operate in it. Obviously, new techniques are needed for this more general case. These techniques were provided by Baillon [1] and simplified by Bruck [2], when the norm is Fréchet-differentiable, a property which is shared by both l_p and L_p spaces for $p \in (1, +\infty)$.

On the other hand, the concept of asymptotically nonexpansive mapping was further extended by Passty [6] to the sequence of mappings which are not necessarily the powers of a given mapping. He has shown that if E has a Fréchet-differentiable norm and if T_n is weakly continuous, then a fixed point of T_n can be obtained by iterating T_n starting at a point of asymptotic regularity.

In this paper, we prove that the sequence

$$x_{n+1} = \alpha_n T_n(x_n) + (1-\alpha_n)x_n \tag{1}$$

of Mann type iteration process converges weakly to some fixed point of T_n. Here T_n is a Passty type asymptotically nonexpansive mapping defined in a uniformly convex Banach space equipped with Fréchet-differentiable norm. We emphasis that no asymptotic regularity condition is posed on T_n. Our result extends and generalizes the results of Passty [6], Xu [8], and others.

2. Preliminaries. Before presenting our main results of this section, we need the following:

Definition 1. A normed space $(E, \|\cdot\|)$ is said to be *uniformly convex* if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $x, y \in E$ with $\|x\|, \|y\| < 1$ and $\|x - y\| \geq \epsilon$, it follows that $\|x + y\| \leq 2(1 - \delta)$.

Definition 2 ([6]). The sequence $\{T_n\}_{n=1}^{\infty}$ of self-mapping of a nonempty subset K of a normed space $(E, \|\cdot\|)$ is said to be *asymptotically nonexpansive* if

$$\|T_n x - T_n y\| \leq k_n \|x - y\| \tag{2}$$

for all x, y in K with $\lim_{n \to \infty} k_n = 1$, where $\{k_n\} \in [1, +\infty)^N$.

For abbreviation, we denote the set of fixed points of T by $\mathrm{Fix}(T)$, the strong convergence by $\longrightarrow$, and the weak convergence by $\xrightarrow{w}$, respectively.

We use the following lemmas to prove our main result.

Lemma 1 ([7, Lem. 1.1]). *Let $(E, \|\cdot\|)$ be a normed space. Let K be a nonempty and bounded subset of E, $\{k_n\} \in [1, +\infty)^N$ with $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ and $T_n : K \longrightarrow K$ be Lipschitzian with respect to k_n for each $n \in N$. Then $\lim_{n \to \infty} \|x_n - x\|$ exists for each $x \in \bigcap_{n \in N} \mathrm{Fix}(T_n)$.*

Lemma 2 ([7, Lem. 1.3]). *Let $(E, \|\cdot\|)$ be a uniformly convex Banach space with Fréchet-differentiable norm. Let K be a nonempty, bounded, closed and convex subset of E, $\{k_n\} \in [1, +\infty)^N$ with $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ and $T_n : K \longrightarrow K$ be Lipschitzian with respect to k_n for each $n \in N$. Suppose that $\{x_n\}$ is given by $x_1 \in K$ and $x_{n+1} = T_n x_n$ for all $n \in N$. Then $\lim_{n \to \infty} J_E(y_1 - y_2)(x_n)$ exists for all $y_1, y_2 \in \bigcap_{n=N} \mathrm{Fix}(T_n)$, where $J_E : E \longrightarrow 2^{E^*}$ denotes the normalized duality mapping, i.e.,*

$$J_E(x) := \{u \in E^* \mid u(x) = \|u\|\|x\| \text{ and } \|u\| = \|x\|\} \tag{3}$$

for all $x \in E$ and, also, $(J_E u, u) = \|u\|^2 = \|J_E u\|^2$ for all $u \in E$.

Now, we give our main result:

Theorem 3. *Let $(E, \|\cdot\|)$ be a uniformly convex Banach space with Fréchet-differentiable norm and K be a nonempty, closed, and convex subset of E. Let F be a subset of K and $S = \{T_n\}_{n=1}^{\infty}$ be an asymptotically nonexpansive sequence of self-mappings of K such that*

$$F \subset \bigcap_{n \in N} \mathrm{Fix}(T_n) \text{ for a sequence } \{k_n\} \in [1, +\infty)^N \quad \text{with } \sum_{n=1}^{\infty}(k_n - 1) < +\infty. \tag{4}$$

Suppose that $\{\alpha_n\} \in [0, 1]$ and $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all $n \in N$ and some $\epsilon > 0$. Assume, also, that there exists a sequence $\{x_n\}$ in K given by $x_1 \in K$ and

$$x_{n+1} = \alpha_n T_n(x_n) + (1 - \alpha_n) x_n \tag{5}$$

for all $n \in N$, for which

$$x_{n_i} \xrightarrow{w} z \text{ implies } z \in F. \tag{6}$$

Then either

(i) *$F = \emptyset$ and $\|x_n\| \longrightarrow +\infty$ or*
(ii) *$F \neq \emptyset$ and $x_n \xrightarrow{w}$ an element of F.*

Proof. Suppose that some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ defined by (5) is bounded. Since E is reflexive (every uniformly convex Banach space is reflexive), the subsequence $\{x_{n_i}\}$ must converge weakly to an element $z \in E$ and, hence, $z \in F$ by (6). Thus, $F = \emptyset$ implies $\|x_n\| \longrightarrow +\infty$.

On the other hand, if $F \neq \emptyset$, then there is some $y_0 \in F$ and, by Lemma 1, $\{\|x_n - y_0\|\}$ is bounded, say, by R. Let $C = \{x \in K \mid \|x - y_0\| \le R\}$. Then C is closed, convex, bounded, and nonempty. Furthermore, $x_n \in C$ for all $n \in N$. In order to apply Lemma 2, we define

$$U_n = \alpha_n T_n + (1-\alpha_n) I \tag{7}$$

for all $n \in N$ where I denotes the identity mapping. Then $U_n(C) \subset C$ for all $n \in N$ because C is convex and $T_n(C) \subset C$. Additionally, we have

$$\begin{aligned} \|U_n x - U_n y\| &\le \alpha_n \|T_n x - T_n y\| + (1-\alpha_n)\|x - y\| \\ &\le \left[\alpha_n k_n + (1-\alpha_n)\right]\|x-y\| \\ &\le k_n \|x - y\| \end{aligned} \tag{8}$$

for all $n \in N$ and $x, y \in C$. Furthermore,

$$x_{n+1} = U_n x_n \tag{9}$$

for all $n \in N$ and

$$\bigcap_{n \in N} \operatorname{Fix}(T_n) = \bigcap_{n \in N} \operatorname{Fix}(U_n) \tag{10}$$

because $\operatorname{Fix}(U_n) = \operatorname{Fix}(T_n)$ for all $n \in N$. Lemma 2 shows that

$$\lim_{n \to \infty} J_E(y_1 - y_2)(x_n) \tag{11}$$

exists for all $y_1, y_2 \in F$ and so, if z_1 and z_2 are two weak subsequential limits of $\{x_n\}$, then $J_E(y_1 - y_2)(z_1 - z_2) = 0$. By (6), z_1 and z_2 are in F. Thus, we may take $y_i = z_i$ for $i = 1, 2$ and so

$$0 = J_E(z_1 - z_2)(z_1 - z_2) = \|z_1 - z_2\|^2. \tag{12}$$

Since all weak subsequential limits of bounded sequence $\{x_n\}$ are, thus, equal, $\{x_n\}$ must converge weakly to an element of F. This completes the proof. □

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