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FIXED POINTS OF ROTATIONS OF n-SPHERE

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ABSTRACT. We show that every rotation of an even-dimensional sphere must have a fixed point.

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The curious "Hairy Ball Theorem" [1] states that *there are no continuous nonvanishing vector fields tangent to the 2k-dimensional sphere* S^{2k}. Hairy Ball Theorem, however, is false for S^{2k-1} (easy to verify), which shows that one can geometrically determine the parity of n in S^n.

Here is another geometric and simpler asymmetry between spheres of odd and even dimensions:

Theorem 1. *Every rotation of* S^{2n} *has at least one fixed point.*

Once again, as an example below illustrates, one can construct rotations of S^{2n-1} that have no fixed point.

Proof. Rotation in $\mathbb{R}^k$ is a linear transformation that preserves distance from the origin. Thus, if A denotes the transformation matrix, then for every $x \in \mathbb{R}^k$,

$$x^T x = (Ax)^T Ax = x^T A^T Ax, \tag{1}$$

which implies that $A^T A = I$ or $A^{-1} = A^T$ (i.e., A is an orthogonal matrix). $A^{-1} = A^T$ implies that $\det(A) = \pm 1$. But rotation is a continuous transformation and hence one can find a continuous chain of matrices $M(t)$ such that $M(0) = I$ and $M(1) = A$ and each $M(t)$, $0 \le t < 1$, represents a rotation. $f(t) = \det(M(t))$ is a continuous function of t with $f(0) = 1$. If $f(1) = -1$, by intermediate value theorem $f(t') = 0$ for $0 < t' < 1$, which contradicts the assumption that $M(t')$ represents a rotation and is therefore nonsingular. Hence, $\det(A) = +1$ (orthogonal matrices with negative determinant represent reflection). $S^{2n} \subset \mathbb{R}^{2n+1}$. Hence, if A represents a rotation in $\mathbb{R}^{2n+1}$, then A is an order $2n+1$ matrix. The characteristic polynomial $P(x) = \det(A - xI)$ is hence of degree $2n+1$. Complex roots of $P(x)$ (if any) occur in conjugate pairs. Hence, $P(x)$ has at least one real root. Further, since the determinant of A is the product of its eigenvalues, the product of the roots of $P(x)$ equals $+1$. The product of a pair of complex conjugates is always nonnegative and hence A must have an even number of negative eigenvalues (counting multiplicity). Since $P(x)$ has $2n+1$ roots in all (counting multiplicity), it has at least one positive eigenvalue, say λ; the eigenvector y of λ

is real.

$$(Ay)^T Ay = \lambda^2 y^T y = y^T y, \tag{2}$$

which implies that $\lambda = +1$ and $Ay = y$. Hence, y is a fixed point of the rotation A. □

Next, consider the following rotation of $S^{2n-1} \subset \mathbb{R}^{2n}$

$$B(\phi_1,\dots,\phi_n) = \begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & & & \\ \sin\phi_1 & \cos\phi_1 & & & \\ & & \ddots & & \\ & & & \cos\phi_n & -\sin\phi_n \\ & & & \sin\phi_n & \cos\phi_n \end{bmatrix} \tag{3}$$

with $0 < \phi_1,\dots,\phi_n < \pi/2$. The eigenvalues of B are $e^{\pm i\phi_1},\dots,e^{\pm i\phi_n}$, none of which is real for $0 < \phi_1,\dots,\phi_n < \pi/2$. Since $+1$ is not an eigenvalue of B, the rotation B cannot have any fixed points.

References

[1] J. Milnor, *Analytic proofs of the "hairy ball theorem" and the Brouwer fixed-point theorem*, Amer. Math. Monthly **85** (1978), 521–524. MR 80m:55001. Zbl 386.55001.

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