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ON THE OSCILLATION OF DELAY DIFFERENTIAL EQUATIONS WITH REAL COEFFICIENTS

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ABSTRACT. A set of necessary conditions and another set of sufficient conditions for the oscillation of all the solutions of

$$\dot{x}(t)+\sum_{i=1}^{n} p_i x(t-\tau_i)=0, \quad \text{where } p_i \in \mathbb{R},\ \tau_i \in \mathbb{R}^+, \quad i=1,2,\dots,n \tag{$*$}$$

are proved.

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1. Introduction. Most of the differential equation models of population dynamics have been derived starting from the following simple format:

$$\frac{dx(t)}{dt}=\{\text{an individual's contribution to population change in unit time}\}x(t), \tag{1.1}$$

where $x(t)$ denotes the density of a population (or biomass) of a single species at time t.

To derive model equations with delays in production and destruction in the following balance equation, assuming there is no immigration or emigration

$$\frac{dx(t)}{dt}=\text{birth rate}-\text{death rate}. \tag{1.2}$$

For instance, if we consider a population of adult flies, then the production or recruitment of adult flies at time t depends on the population of adult at time $t-\tau$, where τ is the time required for the larvae to become adult. If the birth and death rates are governed by density dependent factors, then we have

$$\frac{dx(t)}{dt}=a(x(t-\tau))-b(x(t)), \tag{1.3}$$

where the functions $a(\cdot)$ and $b(\cdot)$ denote density dependent production (recruitment) and destruction (elimination or death) rates, respectively. The oscillation theory of delay differential equations has been extensively developed during the past few years. New applications which involve delay differential equations continue to arise with increasing frequency in the modelling of diverse phenomena in physics, biology, ecology, and physiology. We refer, for example, to the recent books [2, 3, 7, 8] and to the

references cited therein.

Tramov [4] obtained a necessary and sufficient condition for the oscillation of all the solutions of the delay differential equation

$$\dot{x}(t)+\sum_{i=1}^{n} p_i x(t-\tau_i)=0, \tag{1.4}$$

where p_i's are positive real numbers and τ_i's are nonnegative real numbers, $i = 1,2,\dots,n$. It was proved that all the solutions of equation (1.4) are oscillatory if and only if

$$-\lambda+\sum_{i=1}^{n} p_i e^{\lambda\tau_i}>0 \quad \text{for all } \lambda>0. \tag{1.5}$$

The same result was independently discovered and proved by Ladas et al. [5, 6]. Also, Arino and Györi [1] studied equation (1.4) with $p_i \in \mathbb{R}$ and $\tau_i \in \mathbb{R}^+$, $i = 1,2,\dots,n$. It was proved that all the solutions are oscillatory if and only if the characteristic equation

$$\lambda+\sum_{i=1}^{n} p_i e^{-\lambda\tau_i}=0 \tag{1.6}$$

has no real roots.

Györi and Ladas [3] proved that $p\tau > 1/e$ is a necessary and sufficient condition for the oscillation of the solutions of

$$\dot{x}(t)+px(t-\tau)=0, \quad p,\tau\in\mathbb{R}. \tag{1.7}$$

For the equation

$$\dot{x}(t)+px(t-\tau)-qx(t-\sigma)=0, \quad p,q,\tau, \text{ and } \sigma\in\mathbb{R}^+, \tag{1.8a}$$

it was proved that

$$p>q, \qquad \tau\geq\sigma \tag{1.8b}$$

is necessary condition for the oscillation of the solutions of equation (1.8) while

$$p>q, \qquad \tau\geq\sigma, \qquad q(\tau-\sigma)\leq 1, \qquad \text{and} \qquad (p-q)e\tau>\big(1-q(\tau-\sigma)\big) \tag{1.8c}$$

are sufficient conditions for the oscillation of (1.8a).

In this paper, we extend the last results and study equation (1.4) with both positive and negative coefficients in the form

$$\dot{x}(t)+\sum_{i=1}^{n} p_i x(t-\tau_i)=0, \quad p_i\in\mathbb{R}, \tau_i\in\mathbb{R}^+, \tag{1.9}$$

which is an open problem in [3, pp. 56]. At the end of this paper, we give some examples to illustrate the importance and usefulness of the obtained results.

By a solution of equation (1.9) on $[t_0,\infty)$, where $t_0\geq 0$, we mean a continuous function defined on $[t_0-\tau,\infty)$, where $\tau=\max\tau_i$, $i = 1,2,\dots,n$, which is a differentiable function x on $[t_0,\infty)$ and satisfies equation (1.9) for all $t > t_0$.

As it is customary, a solution is called oscillatory if it has arbitrarily large zeros and otherwise it is called nonoscillatory.

2. The main result. Consider equation (1.9) and assume that $p_{k_i} > 0$, $i = 1,2,\dots,\ell$ and that $p_{m_j} \le 0$, $j = 1,2,\dots,r$ with $\ell + r = n$. Let $q_{m_j} = -p_{m_j}$, $j = 1,2,\dots,r$, then the equation takes the form

$$\dot{x} + \sum_{i=1}^{\ell} p_{k_i} x(t-\tau_{k_i}) - \sum_{j=1}^{R} q_{m_j} x\left(t-\tau_{m_j}\right) = 0 \tag{2.1}$$

or, simply,

$$\dot{x} + \sum_{i=1}^{l} p_i x(t-\tau_i) - \sum_{j=1}^{r} q_j x(t-\sigma_j) = 0, \tag{2.2}$$

where p_i, τ_i, q_j, and $\sigma_j \in \mathbb{R}^+, i = 1,2,\dots,\ell, j = 1,2,\dots,r$ with $\ell + r = n, \tau_1 \ge \tau_2 \ge \cdots \ge \tau_\ell$ and $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r$.

Theorem. *Consider the delay differential equation (2.2). Then*

$$\sum_{i=1}^{\ell} p_i > \sum_{j=1}^{r} q_j \qquad \text{and} \qquad \tau_1 \ge \sigma_1 \tag{2.3}$$

are necessary conditions for the oscillation of all the solutions of equation (2.2), while

$$\ell p_i > \sum_{j=1}^{r} q_j \qquad \text{and} \qquad \tau_j \ge \sigma_1 \quad \forall i = 1,2,\dots,\ell, \tag{2.4}$$

$$\sum_{i=1}^{\ell} \left[1 - \sum_{j=1}^{r} q_j(\tau_j - \sigma_j)\right] \ge 0, \tag{2.5}$$

$$\sum_{i=1}^{\ell} \left[\ell p_i - \sum_{j=1}^{r} q_j\right]\tau_i > \frac{1}{e}\left[\sum_{i=1}^{\ell}\left[1 - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)\right]\right] \tag{2.6}$$

are sufficient conditions for oscillation.

Proof. The characteristic equation of equation (2.2) is

$$F(\lambda) = \lambda + \sum_{i=1}^{\ell} p_i e^{-\lambda r_i} - \sum_{j=1}^{r} q_j e^{-\lambda \sigma_j} = 0. \tag{2.7}$$

Assume that all the solutions of equation (2.2) are oscillatory. Consequently, the characteristic equation (2.7) has no real roots. As $F(\infty) = \infty$, it follows that

$$F(0) = \sum_{i=1}^{\ell} p_i - \sum_{j=1}^{r} q_j > 0 \tag{2.8}$$

which implies that

$$\sum_{i=1}^{\ell} p_i > \sum_{j=1}^{r} q_j. \tag{2.9}$$

Also, we have $\tau_1 \geq \sigma_1$ since if $\tau_1 < \sigma_1$, then we get $F(-\infty) = -\infty$, which means that equation (2.7) has a real root. On the other hand, assume that equation (2.2) has nonoscillatory solution and then the characteristic equation (2.7) has a real root λ_0, i.e.,

$$F(\lambda_0) = \lambda_0 + \sum_{i=1}^{\ell} p_i e^{-\lambda_0 \tau_i} - \sum_{j=1}^{r} q_j e^{-\lambda_0 \sigma_j} = 0. \tag{2.10}$$

But $\forall \lambda \in \mathbb{R}$, one can write

$$\lambda\left[1 - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j} \int_0^{\tau_i - \sigma_j} e^{-\lambda s} ds\right] = \left(\lambda - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j}\right) + \left(\sum_{j=1}^{r} q_j\right) e^{-\lambda\tau_i} \quad \forall i = 1,2,\dots,\ell. \tag{2.11}$$

Hence,

$$\sum_{i=1}^{\ell} \lambda\left[1 - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j} \int_0^{\tau_i - \sigma_j} e^{-\lambda s} ds\right] = \ell\left(\lambda - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j}\right) + \left(\sum_{j=1}^{r} q_j\right) \sum_{i=1}^{\ell} e^{-\lambda\tau_i} \tag{2.12}$$

and, consequently, $F(\lambda)$ can be written in the form

$$F(\lambda) = \frac{1}{\ell}\left\{\sum_{i=1}^{\ell} \lambda\left[1 - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j} \int_0^{\tau_i - \sigma_j} e^{-\lambda s} ds\right] - \left(\sum_{j=1}^{r} q_j\right) \sum_{i=1}^{\ell} e^{-\lambda\tau_i}\right\} + \sum_{i=1}^{\ell} p_i e^{-\lambda\tau_i}. \tag{2.13}$$

Then $\forall \lambda \geq 0$, we get

$$F(\lambda) > \frac{1}{\ell}\left\{\sum_{i=1}^{\ell}\left[\ell p_i - \sum_{i=1}^{r} q_j\right] e^{-\lambda\tau_i} + \sum_{i=1}^{\ell} \lambda\left[1 - \sum_{j=1}^{\ell} q_j(\tau_i - \sigma_j)\right]\right\} > 0. \tag{2.14}$$

Consequently, $F(\lambda)$ has no nonnegative real roots and then $\lambda_0 < 0$. Using (2.10) and (2.12), we get

$$\begin{aligned}\sum_{i=1}^{\ell} \lambda_0\left[1 - \sum_{j=1}^{r} q_j e^{-\lambda_0\sigma_j} \int_0^{\tau_i - \sigma_j} e^{-\lambda_0 s} ds\right] &= \ell\left[\lambda_0 - \sum_{j=1} q_j e^{-\lambda_0\sigma_j}\right] + \left(\sum_{i=1}^{\ell} q_j\right) \sum_{i=1}^{\ell} e^{-\lambda_0\tau_i} \\ &= -\sum_{i=1}^{\ell}\left(\ell p_i - \sum_{j=1}^{r} q_j\right) e^{-\lambda_0\tau_i} < 0.\end{aligned} \tag{2.15}$$

Since $\lambda_0 < 0$, from equation (2.15), one can write

$$0 < \sum_{i=1}^{\ell}\left[1 - \sum_{j=1}^{r} q_j e^{-\lambda_0\sigma_j} \int_0^{\tau_i - \sigma_j} e^{-\lambda_0 s} ds\right] < \sum_{i=1}^{\ell}\left[1 - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)\right]. \tag{2.16}$$

Using equations (2.15) and (2.16), we obtain

$$\begin{aligned}\lambda_0 \sum_{i=1}^{\ell}\left[1 - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)\right] &< \sum_{i=1}^{\ell} \lambda_0\left(1 - \sum_{i=1}^{r} q_j e^{-\lambda_0\sigma_j} \int_0^{\tau_i - \sigma_j} e^{-\lambda_0 s} ds\right) \\ &= -\sum_{i=1}^{\ell}\left(\ell p_i - \sum_{j=1}^{r} q_j\right) e^{-\lambda_0\tau_i} < 0,\end{aligned} \tag{2.17}$$

and, consequently,

$$\lambda_0 + \frac{\sum_{i=1}^{\ell}\left(\ell p_i - \sum_{j=1}^{r} q_j\right) e^{-\lambda_0 \tau_i}}{\sum_{i=1}^{\ell}\left(1 - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)\right)} < 0. \tag{2.18}$$

Thus, the equation

$$\lambda + \frac{\sum_{i=1}^{\ell}\left(l p_i - \sum_{j=1}^{r} q_j\right) e^{-\lambda \tau_i}}{\sum_{i=1}^{\ell}\left(1 - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)\right)} = 0 \tag{2.19}$$

has a negative real root, which is a contradiction with (2.6). □

Example 1. Consider the delay differential equation in the form

$$\dot{x}(t) + 4x(t-2.4) + x(t-1.2) - x(t-2.5) - x(t-1.5) = 0. \tag{2.20}$$

This equation has a nonoscillatory solution $x(t) = e^{\lambda t}$, $-13.863 < \lambda < -13.8629$ since $\tau_1 < \sigma_1$.

Example 2. The delay differential equation

$$\dot{x}(t) + x(t-4\pi) - x(t-2\pi) - x\left(t - \frac{3\pi}{2}\right) = 0 \tag{2.21}$$

has an oscillatory solution $x(t) = \sin t$ but not all solutions are oscillatory. It is clear that the necessary conditions are not satisfied. In fact, it has a nonoscillatory solution $x(t) = e^{\lambda t}$, $-0.1026 < \lambda < -0.1025$.

Example 3. For the following delay differential equation

$$\dot{x}(t) + 3x(t-4\pi) - x(t-2\pi) - x\left(t - \frac{3\pi}{2}\right) = 0, \tag{2.22}$$

the necessary conditions are satisfied but the sufficient conditions for oscillations are not satisfied. Actually, it has a nonoscillatory solution $x(t) = e^{\lambda t}, 0.0637239 < \lambda < 0.0637241$.

Example 4. All the solutions of the equation

$$\dot{x}(t) + x(t-2.7) + 4x(t-2.6) - x(t-2.5) - 0.6x(t-1.5) = 0 \tag{2.23}$$

are oscillatory since all the necessary and sufficient conditions are satisfied.

References

[1] O. Arino and I. Györi, *Necessary and sufficient condition for oscillation of a neutral differential system with several delays*, J. Differential Equations **81** (1989), no. 1, 98–105. MR 90i:34101. Zbl 691.34054.

[2] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Mathematics and its Applications, vol. 74, Kluwer Academic Publishers Group, Dordrecht, 1992. MR 93c:34150. Zbl 752.34039.

[3] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations: With Applications*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1991. MR 93m:34109. Zbl 780.34048.

[4] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, vol. 191, Academic Press, Inc., Boston, MA, 1993. MR 94f:34001. Zbl 777.34002.

[5] G. Ladas, Y. G. Sficas, and I. P. Stavroulakis, *Necessary and sufficient conditions for oscillations*, Amer. Math. Monthly **90** (1983), no. 9, 637–640. MR 84j:34082. Zbl 526.34054.

[6] ———, *Necessary and sufficient conditions for oscillations of higher order delay differential equations*, Trans. Amer. Math. Soc. **285** (1984), no. 1, 81–90. MR 85j:34146. Zbl 537.34069.

[7] G. S. Ladde, V. Lakshmikantham, and B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 110, Marcel Dekker, Inc., New York, 1987. MR 90h:34118. Zbl 832.34071.

[8] M. I. Tramov, *Conditions for the oscillation of the solutions of first order differential equations with retarded argument*, Izv. Vyssh. Uchebn. Zaved. Mat. **154** (1975), no. 3, 92–96 (Russian). MR 52 961. Zbl 319.34070.

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