THE UNIVERSAL SEMILATTICE COMPACTIFICATION OF A SEMIGROUP

H. R. EBRAHIMI VISHKI and M. A. POURABDOLLAH

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ABSTRACT. The universal abelian, band, and semilattice compactifications of a semitopological semigroup are characterized in terms of three function algebras. Some relationships among these function algebras and some well-known ones, from the universal compactification point of view, are also discussed.

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1. Introduction. The notion of semigroup compactification has been produced in several principal ways, in whose main approach the Gelfand-Naimark theory of commutative C^* -algebras is employed. In fact, the spectrum of every m -admissible algebra of functions is a semigroup compactification. Moreover, some of these compactifications enjoy a universal property P . For instance, De Leeuw and Glicksberg in their influential paper [2], characterized the universal property of (weakly) almost periodic compactification. The existence of the universal P -compactification (using the subdirect product methods) for a broad variety of properties P , is guaranteed by Junghenn and Pandian [7]. The construction of some of the better known universal P -compactifications in terms of m -admissible algebras of functions are collected in Berglund et al. [1], which is our ground reference. The universal right simple, left simple, and group compactifications are characterized in terms of some types of distal functions [6]. In two recent papers [9, 10], Pandian has examined the universal mapping property of generalized distal, and quasiminimal distal functions. Also, in an earlier paper [3], we have characterized the universal nilpotent group compactification. The present paper deals with the construction of three m -admissible algebras AB , BD , and SL , which characterize the universal abelian, band, and semilattice compactifications of a semitopological semigroup.

2. Preliminaries. For background and notations we follow Berglund et al. [1] as much as possible. In what follows, S is a semitopological semigroup unless otherwise stipulated. A (semigroup) compactification of S is a pair (ψ, X) , where X is compact, Hausdorff, right topological semigroup and $\psi : S \rightarrow X$ is a continuous homomorphism with dense image such that, for all $s \in S$, the mapping $x \mapsto \psi(s)x : X \rightarrow X$ is continuous.

The C^* -algebra of all continuous bounded complex-valued functions on a topological space Y is denoted by $C(Y)$. For $C(S)$ left and right translations, L_s and R_t , are

defined for all $s, t \in S$ by $(L_s f)(t) = f(st) = (R_t f)(s)$, $f \in C(S)$. A left translation invariant C^* -subalgebra F of $C(S)$ (i.e., $L_s f \in F$ for all $s \in S$ and $f \in F$), containing the constant functions, is called m -admissible if the function $s \mapsto (T_\mu f)(s) = \mu(L_s f)$ is in F for all $f \in F$ and $\mu \in S^F$ ($=$ the spectrum of F). If so, S^F under the multiplication $\mu\nu = \mu \circ T_\nu$ ($\mu, \nu \in S^F$), furnished with the Gelfand topology, makes (ε, S^F) a compactification (called the F -compactification) of S , where $\varepsilon : S \rightarrow S^F$ is the evaluation mapping. Conversely, if (ψ, X) is a compactification of S , then $\psi^*(C(X))$ is an m -admissible subalgebra of $C(S)$, where ψ^* is the dual mapping of ψ , and this correspondence between compactifications of S and m -admissible subalgebras of $C(S)$ is one-to-one (see [1, Thm. 3.1.7]).

A compactification (ψ, X) of S , possessing a certain property P , is called the universal P -compactification if for any other compactification (φ, Z) , having the property P , there exists a homomorphism $\pi : (\psi, X) \rightarrow (\varphi, Z)$, where π is a continuous mapping from X into Z with $\pi \circ \psi = \varphi$, or equivalently, $\varphi^*(C(Z)) \subseteq \psi^*(C(X))$ (see [1, Thm. 3.1.9]).

Some of the usual m -admissible subalgebras of $C(S)$, that are needed in the sequel, are the left multiplicatively continuous, weakly almost periodic, almost periodic, strongly almost periodic, distal, minimal distal, and strongly distal functions on S . These are denoted by LMC, WAP, AP, SAP, D, MD and SD , respectively. We also write GP for $MD \cap SD$, LZ for $\{f \in C(S) : f(st) = f(s) \text{ for all } s, t \in S\}$, and RZ for $\{f \in C(S) : f(st) = f(t) \text{ for all } s, t \in S\}$. Here, and also for other emerging spaces, when there is no risk of confusion, we have suppressed the letter S from the notation. For ease of reference, we mention the next proposition which describes the universal mapping properties of these m -admissible algebras.

PROPOSITION 2.1. *See [1, Chap. 4] and [6, Thm. 3.4]. The $LMC, WAP, AP, SAP, D, MD, SD, GP, LZ$, and RZ -compactifications are universal with respect to the properties of being a (right topological) semigroup, a semitopological semigroup, a topological semigroup, a topological group, an inflation of a rectangular group, a left simple semigroup, a right simple semigroup, a group, a left zero semigroup, and a right zero semigroup, respectively.*

3. The main results. To follow the main objective, we examine the properties of AB and BD , where

$$AB = \{f \in WAP : f(st) = f(ts), \text{ and } f(stu) = f(sut) \text{ for all } s, t, u \in S\} \quad (3.1)$$

and BD consists of those $f \in LMC$ such that

$$\begin{aligned} \lim_{\alpha} \left(\lim_{\alpha} R_{s_{\alpha}} f \right) (s_{\alpha}) &= \lim_{\alpha} f(s_{\alpha}); \\ \lim_{\alpha} \left(\lim_{\alpha} R_{s_{\alpha}} \left(\lim_{\alpha} R_{t_{\alpha}} f \right) \right) (s_{\alpha}) &= \lim_{\alpha} \left(\lim_{\alpha} R_{t_{\alpha}} f \right) (s_{\alpha}); \\ \lim_{\alpha} R_{s_{\alpha}} \left(\lim_{\alpha} R_{s_{\alpha}} f \right) &= \lim_{\alpha} R_{s_{\alpha}} f; \\ \lim_{\alpha} R_{s_{\alpha}} \left(\lim_{\alpha} R_{s_{\alpha}} \left(\lim_{\alpha} R_{t_{\alpha}} f \right) \right) &= \lim_{\alpha} R_{s_{\alpha}} \left(\lim_{\alpha} R_{t_{\alpha}} f \right) \end{aligned} \quad (3.2)$$

for all nets $\{s_{\alpha}\}$ and $\{t_{\alpha}\}$ in S for which the relevant pointwise limits exist.

Also, we write SL for $AB \cap BD$. The next lemma, which requires a routine proof, characterizes AB and BD in terms of the elements of S^{WAP} and S^{LMC} , respectively.

LEMMA 3.1. (i) *A function $f \in WAP$ is in AB if and only if $\mu\nu(f) = \nu\mu(f)$ and $T_{\mu\nu}f = T_{\nu\mu}f$ for all $\mu, \nu \in S^{WAP}$.*

(ii) *A function $f \in LMC$ is in BD if and only if $\mu^2(f) = \mu(f)$, $\mu^2\nu(f) = \mu\nu(f)$, $T_{\mu^2}f = T_{\mu}f$, and $T_{\mu^2\nu}f = T_{\mu\nu}f$ for all $\mu, \nu \in S^{LMC}$.*

The following theorem states the main properties of AB, BD , and SL .

THEOREM 3.2. *AB, BD , and SL are those m -admissible subalgebras of $C(S)$, whose corresponding compactifications of S are universal with respect to the properties of being an abelian semigroup, a band, and a semilattice, respectively.*

PROOF. It is enough to prove the conclusion for AB and BD . Using Lemma 3.1, the m -admissibility of AB and BD can be easily demonstrated, and also it follows that S^{AB} and S^{BD} are abelian and a band, respectively. Let (ψ, X) be an abelian compactification of S , then $C(X) = AB(X)$ and so $\psi^*(C(X)) = \psi^*(AB(X)) \subseteq AB(S)$, where the latter inclusion can be easily verified. Thus, (ε, S^{AB}) is the universal abelian compactification of S . Similarly, to see that (ε, S^{BD}) is universal with respect to the property of being a band, it is enough to show that for any other band compactification (φ, Z) of S , $\varphi^*(C(Z)) \subseteq BD(S)$. For this, let $\pi : (\varepsilon, S^{LMC}) \rightarrow (\varphi, Z)$ be the canonical homomorphism whose existence is guaranteed by the universal property of (ε, S^{LMC}) . If $g \in C(Z)$, then $\varphi^*(g) \in LMC(S)$ and for all $\mu \in S^{LMC}$, $\mu^2(\varphi^*(g)) = g(\pi(\mu)^2) = g(\pi(\mu)) = \mu(\varphi^*(g))$. A similar argument shows that, for each $\nu \in S^{LMC}$, $\mu^2\nu(\varphi^*(g)) = \mu\nu(\varphi^*(g))$, $T_{\mu^2}\varphi^*(g) = T_{\mu}\varphi^*(g)$, and $T_{\mu^2\nu}\varphi^*(g) = T_{\mu\nu}\varphi^*(g)$. Now, Lemma 3.1 shows that $\varphi^*(g) \in BD(S)$, as required. $\square$

It is trivial that $BD \subseteq BD_c$ (with the equality holding in the compact case), where

$$BD_c = \{f \in C(S) : f(s^2) = f(s), f(s^2t) = f(st) = f(st^2), \\ \text{and } f(st^2u) = f(stu) \text{ for all } s, t, u \in S\}. \quad (3.3)$$

The joint continuity of the multiplication of S^{AP} implies that $BD \cap AP = BD_c \cap AP$. Furthermore, S^{SL} is a compact semitopological semilattice, so by Lawson's (joint continuity) theorem [8], $SL \subseteq AP$. Thus, $SL = AP \cap BD_c \cap AB$; more precisely:

PROPOSITION 3.3. *$SL = \{f \in AP : f(s^2) = f(s), f(s^2t) = f(st) = f(ts), \text{ and } f(st^2u) = f(stu) = f(sut), \text{ for all } s, t, u \in S\}$.*

The universal properties of (ε, S^{BD}) and $(\varepsilon, S^{BD \cap D})$ imply that $(\varepsilon, S^{BD \cap D})$ is universal with respect to the property of being a rectangular band [1, Exercise 1.1.48]. Furthermore, since every such rectangular band is a topological semigroup, $BD \cap D \subseteq AP$ which implies that $BD \cap D = BD_c \cap D \cap AP$. On the other hand, an adaptation of Junghenn's ideas in the proof of Proposition 3.10 of [6] implies that $BD \cap D = \langle LZ \cup RZ \rangle = LZ \otimes RZ$, where $\langle LZ \cup RZ \rangle$ is the C^* -subalgebra of $C(S)$ generated by $LZ \cup RZ$ and $LZ \otimes RZ$ is the topological tensor product of LZ and RZ ; i.e., the completion in the least cross norm of the algebraic tensor product.

As a consequence of the universal properties of (ε, S^{GP}) and (ε, S^{AB}) , it is trivial that $(\varepsilon, S^{AB \cap GP})$ is the universal abelian group compactification of S . Some other facts about $AB \cap GP$ are collected in the next result. Also, see [3].

PROPOSITION 3.4. (i) $AB \cap MD = AB \cap GP = AB \cap SD = \{f \in SAP : f(stu) = f(sut), \text{ for all } s, t, u \in S\}$.

(ii) $AB \cap GP$ is the closed linear span of the set of all continuous characters of S .

PROOF. The facts that $S^{AB \cap MD}$ and $S^{AB \cap SD}$ are abelian groups and that $(\varepsilon, S^{AB \cap GP})$ is universal with respect to the property of being an abelian group imply that $AB \cap MD = AB \cap GP = AB \cap SD \subseteq SAP$, where the latter containment is obtained from the Ellis' (joint continuity) theorem [4]. Furthermore, the other condition in the definition of AB , i.e., $f(st) = f(ts)$ is automatically deduced from $f(stu) = f(sut)$ and the fact that $f \in SAP$. The observation that the dual mapping of ε from $C(S^{AB \cap GP})$ onto $AB \cap GP$ establishes a one-to-one correspondence between the continuous characters of $S^{AB \cap GP}$ and those of S and using the Peter-Weyl theorem, [5, Thm. 22.17], for $C(S^{AB \cap GP})$ imply that $AB \cap GP$ is the closed linear span of the continuous characters of S . $\square$

EXAMPLES AND REMARKS 3.5.

(i) For all right zero and left zero semigroups, it is simple to verify that $AB = \mathbb{C}$ (i.e., consists of the constant functions only) and that $BD = C(S)$. Also, for all groups $BD = \mathbb{C}$.

(ii) Consider the discrete semigroup $S = \{a, b, c, d\}$, with multiplication given by: a as a left identity, b and c be as left zeros, and $ds = c$ for all $s \in S$ (see [1, 1.1.7]). A direct computation shows that $AB = \{f \in C(S) : f(b) = f(c) = f(d)\}$ and $BD = \{f \in C(S) : f(c) = f(d)\}$.

(iii) Let $S_3 = \langle a, b \mid a^3 = b^2 = (ab)^2 = 1 \rangle$ be the symmetric group of order 6. One may directly show that $AB(S_3) = \{f \in C(S_3) : f(1) = f(a) = f(a^2), \text{ and } f(b) = f(ab) = f(a^2b)\}$. Of course, $BD(S_3) = \mathbb{C}$.

(iv) An inductive proof shows that a function $f \in WAP$ lies in AB if and only if

$$f(\text{each finite product of elements of } S) = f(\text{each re-ordering of it}).$$

(v) Similar to what we have preceding to Proposition 3.3, using the Lawson's theorem, [8], one may show that for abelian semigroups $BD \cap WAP = SL = BD \cap AP$. Thus, for semilattices, $SL = AP$.

(vi) The equality $BD \cap MD = LZ$ can be easily demonstrated from the fact that all left simple bands are left zero semigroups. Similarly, $BD \cap SD = RZ$. Also, we trivially have $BD \cap GP = BD \cap SAP = \mathbb{C}$.

(vii) The invariant mean on the abelian semigroup S^{AB} induces a unique invariant mean on AB , where the uniqueness is obtained from the fact that the m -admissible subalgebras of WAP cannot have more than one invariant mean (see [1, Cor. 2.3.28, Exereise 4.2.7]). A similar statement holds for SL and $AB \cap GP$. But BD , in general, is not even left amenable. For example, for $S = \{a, b, c, d\}$ as in part (ii), let f in BD be such that $f(b) \neq f(c)$, then for each left invariant mean m on BD , $f(b) = m(L_b f) = m(L_c f) = f(c)$ and this contradicts the choice of f .

(viii) It should be mentioned that $AB, SL, BD \cap D$, and $AB \cap GP$ are also admissible, i.e., they are invariant under T_μ for all μ in their duals [1, Cor. 4.2.7]. But we guess that BD is not admissible in general. It would be desirable to investigate the inclusion $BD \subseteq WAP$.

(ix) Parallel to BD and also SL which are defined by right translates, we have the analogous spaces defined by left translates. It is a matter of fact that the left and right notations do not change the structure of SL (see Proposition 3.3). A natural question that arises is whether they do not change BD . In our opinion, there is a close tie between the latter question and the inclusion $BD \subseteq WAP$. See (viii).

(x) It is obvious that the SL -compactification of the direct product of two semitopological semigroups is isomorphic (in the sense of [1, Sec. 5.2]) to the direct product of their SL -compactifications. A similar fact holds for the $(AB \cap GP)$ -compactification; (more generally, $(AB \cap GP)$ -compactification, roughly speaking, passes through semidirect products. See [1, Lem. 5.2.3]).

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