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DOUBLE MULTIPLIERS ON TOPOLOGICAL ALGEBRAS

L. A. KHAN, N. MOHAMMAD, and A. B. THAHEEM

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ABSTRACT. The main purpose of this paper is to investigate some topological properties of the double multiplier algebra on a topological algebra. Let $M_d(A)$ be the double multiplier algebra on a topological algebra A, and let u and s be the uniform and strong operator topologies on $M_d(A)$, respectively. It is shown, under some additional hypotheses on A, that

(1) $M_d(A)$ is u- and s-complete;
(2) A is a u-closed two-sided ideal in $M_d(A)$;
(3) A is s-dense in $M_d(A)$;
(4) s and u have the same bounded sets;
(5) each continuous onto homomorphism $\phi : A \to B$ has a unique extension $\tilde{\phi} : (M_d(A), s_A) \to (M_d(B), s_B)$.

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1. Introduction. The theory of double multipliers (i.e., of double centralizers) was developed for topological algebras by Johnson [7] and further investigated in the case of Banach algebras and C^*-algebras by Busby [2], Fontenot [5], Taylor [17], Tomiuk [18], Argün and Rowlands [1], and others; see the monographs [9, 12] for additional references. If A is a commutative C^*-algebra, that is $A = C_0(X)-$ the algebra of all complex-valued continuous functions which vanish at infinity on a locally compact Hausdorff space $X-$, then $M_d(A)$, the algebra of all double multipliers of A, is $C_b(X)-$ the algebra of all complex-valued bounded continuous functions on X [21]. The noncommutative generalization of the relationship between $C_0(X)$ and $C(X)$ was found to be useful in the work of Busby [2], Taylor [17], Davenport [4], and Lazar and Taylor [12].

In view of the applications of (nonnormed) topological algebras in other fields such as quantum mechanics and quantum statistics (see, e.g., Lassner [10, 11]) and recent developments in the theory of topological algebras (see, for instance, the book of Mallios [13]), it is important to consider operators on more general classes of topological algebras. More recently, Phillips [14, 15] has studied inner and approximately inner derivations on pro-C^*-algebras (inverse limits of C^*-algebras, also called LMC^*-algebras) using multipliers, while Van Daele [20] has considered multipliers on Hopf algebras which provide a natural framework to study quantum groups. Therefore, it is important to develop the theory of multipliers for general topological algebras and, in particular, for metrizable topological algebras.

In this paper, we are mainly concerned with the linear topological properties of

double multiplier algebra $M_d(A)$ endowed with the uniform and strong operator topologies u and s, respectively. In Section 2, we define multipliers and double multipliers on an algebra A and summarize some basic results for later use. In Section 3, we introduce the u and s topologies on $M_d(A)$ and study their properties. Our main results include extensions of some results of Busby [2] and Phillips [14] from C^*- and $\sigma - C^*$-algebras to more general classes of topological algebras.

2. Preliminaries. Let A be a complex Hausdorff topological algebra in which multiplication is associative and separately continuous. An algebra A is said to be *proper* (or *without order*) if $aA = Aa = \{0\}$ implies that $a = 0$. We note that A is proper in each of the following cases:

(i) A has an identity;
(ii) A is a topological algebra with an approximate identity (e.g., A is a B^*-algebra);
(iii) A is a topological algebra with an orthogonal basis [5].

For the general theory of topological vector spaces and topological algebras, we refer to the books of Rudin [16] and Mallios [13]. A mapping $T : A \to A$ is called a *multiplier* (respectively, left multiplier, right multiplier) on A if $aT(b) = T(a)b$ (respectively $T(ab) = T(a)b$, $T(ab) = aT(b)$) for all $a, b \in A$ (see, e.g., [6]). A pair (S, T) of mappings $S, T : A \to A$ is called a *double multiplier* (or a *double centralizer*) on A if $aS(b) = T(a)b$ for all $a, b \in A$. Let $M(A)$ denote the set of all continuous multipliers on A, and let $M_d(A)$ denote the set of all double multipliers (S, T) of A with S and T continuous. For convenience, we summarize some basic properties of these multipliers in the following theorems:

Theorem 2.1. *Let A be a proper topological algebra. Then*

(a) *If $(S, T) \in M_d(A)$, then S is a left multiplier and T is a right multiplier on A.*
(b) *Each $T \in M(A)$ is linear; if $(S, T) \in M_d(A)$, then S and T are linear.*
(c) *$M(A)$ is a commutative algebra with composition as multiplication (i.e., $(T_1T_2)(a) = T_1(T_2(a))$) and it has the identity $I : A \to A$, $I(a) = a$.*
(d) *$M_d(A)$ is an algebra with identity (I, I) under the operations*

$$(S, T) + (S_1, T_1) = (S + S_1, T + T_1), \tag{2.1}$$

$$\lambda(S, T) = (\lambda S, \lambda T), \tag{2.2}$$

and

$$(S, T)(S_1, T_1) = (SS_1, T_1T), \quad \lambda \in \mathbb{C}. \tag{2.3}$$

(e) *If A is commutative, then $M_d(A)$ is commutative and $M_d(A) \equiv M(A)$; in fact, if $(S, T) \in M_d(A)$, then $S = T$.*
(f) *If A is a Banach algebra, then so is $M_d(A)$ with the norm given by $\|(S, T)\| = \max\{\|S\|, \|T\|\}$.*

Proof. See [2, 7]. □

For any $a \in A$, let L_a, $R_a : A \to A$ be given by $L_a(x) = ax$ and $R_a(x) = xa$, $x \in A$. Clearly, $(L_a, R_a) \in M_d(A)$. It is easy to see that, for any $a \in A$ and $(S, T) \in M_d(A)$,

$$(L_a, R_a)(S, T) = (L_{T(a)}, R_{T(a)}), \qquad (S, T)(L_a, R_a) = (L_{S(a)}, R_{S(a)}). \tag{2.4}$$

We define a map $\mu : A \to M_d(A)$ by $\mu(a) = (L_a, R_a)$, $a \in A$.

Theorem 2.2. *Let A be a Hausdorff topological algebra. Then*
(a) *μ is linear, algebra homomorphism, and continuous.*
(b) *μ is one-one if and only if A is proper.*
(c) *μ is onto if and only if A has identity.*
(d) *If A is proper, then $\mu(A)$ is a two-sided ideal in $M_d(A)$.*

Proof. See [2, 7]. □

The following result, due to Johnson [7, 8] (see also Wang [21, p. 1132]), is concerned with the linearity and continuity of multipliers on A. For completeness, we include its proof here.

Theorem 2.3. (a) *Suppose that A is a complete metrizable LMC algebra with a uniformly bounded left (right) approximate identity. If T is a left (right) multiplier on A, then T is linear and continuous.*
(b) *Suppose that A is a proper complete metrizable algebra. If (S,T) is a double multiplier on A, then S and T are linear and continuous.*

Proof. (a) Let T be a left multiplier on A. By a generalization of the Cohen's factorization theorem (see [3]), given any sequence $\{a_n\} \subseteq A$ with $a_n \to 0$, there exist $b \in A$ and $\{c_n\} \subseteq A$ with $c_n \to 0$ such that $a_n = bc_n$ for all $n \geq 1$. To show that T is linear, let $a_1, a_2 \in A$ and $\alpha, \beta \in \mathbb{C}$. Taking $\{a_n\} = \{a_1, a_2, 0, 0, \dots\}$, there exist $b, c_1, c_2 \in A$ such that $a_1 = bc_1$ and $a_2 = bc_2$. So,

$$\begin{aligned} T(\alpha a_1 + \beta a_2) &= T(b(\alpha c_1 + \beta c_2)) = T(b)(\alpha c_1 + \beta c_2) \\ &= \alpha T(bc_1) + \beta T(bc_2) = \alpha T(a_1) + \beta T(a_2). \end{aligned} \tag{2.5}$$

To show that T is continuous, let $\{a_n\} \subseteq A$ with $a_n \to 0$. We can write $a_n = bc_n$, where $b \in A$ and $\{c_n\} \subseteq A$ with $c_n \to 0$. Hence, $T(a_n) = T(bc_n) = T(b)c_n \to 0$. In the case of a right multiplier, the proof is similar to the above.

(b) Let $(S,T) \in M_d(A)$. Then S and T are linear by Theorem 2.1(b). In view of the closed graph theorem [16, Thm. 2.15], it suffices to show that S and T have closed graphs. Let $\{a_\alpha\}$ be a net in A with $a_\alpha \to a \in A$ and $S(a_\alpha) \to b \in A$. Since multiplication is separately continuous, for any $x \in A$, $xa_\alpha \to xa$ and $xS(a_\alpha) \to xb$. Hence,

$$xS(a) = T(x)a = \lim_\alpha T(x)a_a = \lim_\alpha xS(a_\alpha) = xb. \tag{2.6}$$

Since A is proper, we have $S(a) = b$. Hence, S has a closed graph. Similarly, T also has a closed graph. □

3. Main results. In the sequel, A denotes a proper Hausdorff topological algebra with multiplication jointly continuous. Following Johnson [7], the *uniform operator topology* u (respectively, the *strong operator topology* s) on $M_d(A)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all the sets of the form

$$N(D,W) = \{(S,T) \in M_d(A) : S(D) \subseteq W \text{ and } T(D) \subseteq W\}, \tag{3.1}$$

where D is a bounded (respectively, finite) subset of A and W is a neighborhood of 0 in A. (The topology s is sometimes called the *strict topology*; see, e.g., [1, 2, 4, 5, 14, 17, 18, 21]). Clearly, $s \le u$. It is easy to see that $M_d(A)$, endowed with each of u and s, is a topological algebra in which multiplication is separately continuous. In [7], Johnson observed that if A is a locally convex barrelled quasi-complete metrizable algebra, then $(M_d(A), u)$ and $(M_d(A), s)$ are quasi-complete. In this section, we consider the completeness and some other properties of $(M_d(A), u)$ and $(M_d(A), s)$ without the local convexity assumption on A. We also consider some conditions on A under which A is u-closed with respect to the uniform topology (that is, u-closed) and dense with respect to the strong operator topology (that is, s-dense) in $M_d(A)$.

The following result extends [2, Thm. 2.11 and Prop. 3.1] to topological algebras.

Theorem 3.1. (a) *If A is complete and metrizable, then $(M_d(A), u)$ is complete.*

(b) *If A is complete, then A is a u-closed two-sided ideal in $M_d(A)$, under the identification $\mu : a \to (L_a, R_a)$.*

Proof. (a) Suppose that A is complete and metrizable and $\{(S_\alpha, T_\alpha)\}$be a Cauchy net in $(M_d(A), u)$. Then it easily follows that, for each $a \in A$, $\{S_\alpha(a)\}$ and $\{T_\alpha(a)\}$ are Cauchy nets in A. Consequently, the mappings $S, T : A \to A$, given by $S(a) = \lim_\alpha S_\alpha(a)$ and $T(a) = \lim_\alpha T_\alpha(a) (a \in A)$, are well-defined. Further, for any $a, b \in A$,

$$aS(b) = \lim_\alpha aS_\alpha(b) = \lim_\alpha T_\alpha(a)b = T(a)b. \tag{3.2}$$

Hence, by Theorem 2.3(b), $(S, T) \in M_d(A)$. We now show that $(S_\alpha, T_\alpha) \xrightarrow{u} (S, T)$. Let D be a bounded subset of A and let W be a closed neighborhood of 0 in A. There exists an index α_0 such that

$$S_\alpha(a) - S_\beta(a) \in W \qquad \text{and} \qquad T_\alpha(a) - T_\beta(a) \in W \tag{3.3}$$

for all $a \in D$ and $\alpha, \beta \ge \alpha_0$. Since W is closed, fixing $\alpha \ge \alpha_0$ and taking $\lim_\beta$, we get

$$S_\alpha(a) - S(a) \in W \qquad \text{and} \qquad T_\alpha(a) - T(a) \in W \tag{3.4}$$

for all $a \in D$. Hence, for any $\alpha \ge \alpha_0$,

$$(S_\alpha, T_\alpha) - (S, T) \in N(D, W). \tag{3.5}$$

Thus, $(M_d(A), u)$ is complete.

(b) Suppose that A is complete. We have already seen in Theorem 2.2 that $\mu(A)$ is a two-sided ideal in $M_d(A)$. To show that $\mu(A)$ is u-closed in $M_d(A)$, let $(S, T) \in M_d(A)$ with $(S, T) \in \overline{\mu(A)}^u$. There exists a net $\{a_\alpha \subseteq A\}$ such that $(L_{a_\alpha}, R_{a_\alpha}) \xrightarrow{u} (S, T)$. Then, for any $b \in A$, $a_\alpha b \to S(b)$ and $ba_\alpha \to T(b)$, and so $\{a_\alpha b\}$ and $\{ba_\alpha\}$ are Cauchy nets in A. Since A is proper, it follows easily that $\{a_\alpha\}$ is a Cauchy net in A. Since A is complete, $a_\alpha \to a \in A$. Hence, by continuity of μ,

$$(S, T) = \lim_\alpha \mu(a_\alpha) = \mu(a) = (L_a, R_a), \tag{3.6}$$

and so $(S, T) \in \mu(A)$. Thus, $\mu(A)$ is u-closed in $M_d(A)$. □

The following result was proved in [2] for A, a C^*-algebra and in [14] for A, a pro-C^*-algebra.

Theorem 3.2. (a) *If A is complete and metrizable, then $(M_d(A), s)$ is complete.*

(b) *If A is complete and has a two-sided approximate identity (not necessarily bounded), then A is s-dense in $M_d(A)$.*

Proof. (a) Suppose that A is complete and metrizable, and let $\{(S_\alpha, T_\alpha)\}$ be a Cauchy net in $(M_d(A), s)$. Then, for any $a \in A$, $\{S_\alpha(a)\}$ and $\{T_\alpha(a)\}$ are Cauchy nets in A. Consequently, the mappings $S, T : A \to A$ given by $S(a) = \lim_\alpha S_\alpha(a)$ and $T(a) = \lim_\alpha T_\alpha(a)$ $(a \in A)$ are well-defined and linear. Further, for any $a, b \in A$,

$$aS(b) = \lim_\alpha aS_\alpha(b) = \lim_\alpha T_\alpha(a)b = T(a)b, \tag{3.7}$$

and so by Theorem 2.3(b), $(S, T) \in M_d(A)$. To show that $(S_\alpha, T_\alpha) \xrightarrow{s} (S, T)$, let D be a finite subset of A and let W be a neighborhood of 0 in A. There exists an index α_0 such that

$$S_\alpha(a) - S(a) \in W \qquad \text{and} \qquad T_\alpha(a) - T(a) \in W \tag{3.8}$$

for all $\alpha \geq \alpha_0$ and all $a \in D$ (since D is finite). Hence, $(S_\alpha, T_\alpha) - (S, T) \in N(D, W)$ for all $\alpha \geq \alpha_0$. Thus, $(M_d(A), s)$ is complete.

(b) Suppose that A is complete, and let $\{e_\lambda : \lambda \in I\}$ be a two-sided approximate identity for A. We need to show that $\mu(A)$ is s-dense in $M_d(A)$. Let $(S, T) \in M_d(A)$, and let D be a finite subset of A and W a neighborhood of 0 in A. We claim that, for some $\lambda \in I$, $\mu(T(e_\lambda)) - (S, T) \in N(D, W)$. Now, by definition, $e_\lambda b \to b$ and $be_\lambda \to b$ for all $b \in A$. Since D is finite, we can choose a $\lambda_0 \in I$ such that

$$e_\lambda S(a) - S(a) \in W \qquad \text{and} \qquad T(ae_\lambda) - T(a) \in W \tag{3.9}$$

for all $a \in D$ and all $\lambda \geq \lambda_0$. Then, for any $a \in D$ and $\lambda \geq \lambda_0$,

$$L_{T(e_\lambda)}(a) - S(a) = T(e_\lambda)a - S(a) = e_\lambda S(a) - S(a) \in W \tag{3.10}$$

and

$$L_{T(e_\lambda)}(a) - T(a) = aT(e_\lambda) - T(a) = T(ae_\lambda) - T(a) \in W. \tag{3.11}$$

Thus, $\mu(A)$ is s-dense in $M_d(A)$ and this completes the proof. □

As an application, we get an extension of Tomiuk's result [18, Lem. 2.1].

Corollary 3.3. *Suppose that A is complete and has two-sided approximate identity, and let J be a two-sided ideal in A. Then J is u-dense in A if and only if J is s-dense in $M_d(A)$.*

Proof. Suppose that $\overline{J}^u = A$. We need to show that $\overline{\mu(J)}^s = M_d(A)$. Since $s \leq u$; $\overline{J}^u \subseteq \overline{J}^s$; hence $A \subseteq \overline{J}^s$. By Theorem 3.2(b), $\overline{\mu(A)}^s = M_d(A)$, and so $M_d(A) \subseteq \overline{\mu(J)}^s$. Conversely, suppose that $\overline{\mu(J)}^s = M_d(A)$. Let $\{e_\lambda : \lambda \in I\}$ be a two-sided approximate identity for A, and let $a \in A$. Since $\mu(A) \subseteq M_d(A) = \overline{\mu(J)}^s$, $\mu(a) \in \overline{\mu(J)}^s$ and so there

exists a net $\{a_\alpha\} \subseteq J$ such that $\mu(a_\alpha) \xrightarrow{s} \mu(a)$. Then, for each $\lambda \in I$, $\lim_\alpha \mu(a_\alpha e_\lambda) - \mu(ae_\lambda) = 0$. Now, $a_\alpha e_\lambda \in J$ and so $\mu(a_\alpha e_\lambda) \in \overline{\mu(J)}^s$ for each $\lambda \in I$. Consequently, $a \in \overline{J}^u$, which shows that $\overline{J}^u = A$. □

A topological vector space (TVS) E is called *ultrabornological* if every bounded linear map from E into any TVS is continuous. It follows from [16, Thm. 1.32] that every metrizable TVS is ultrabornological.

Theorem 3.4. *Suppose that A is complete and metrizable. Then*
(a) *s and u have the same bounded sets.*
(b) *If $(M_d(A), s)$ is ultrabornological, then $s = u$ on $M_d(A)$.*

Proof. (a) Since $s \le u$, every u-bounded set in $M_d(A)$ is bounded. Conversely, let H be any s-bounded set in $M_d(A)$. Then, if $a \in A$, for each neighborhood V of 0 in A, there exists a constant $r_a > 0$ such that $H \subseteq r_a N(\{a\}, V)$. This implies that, for each $a \in A$, $\{(S(a), T(a)) : (S,T) \in H\}$ is a bounded set in $A \times A$. By the principle of uniform boundedness [16, Thm. 2.5], the collection $\{S, T : (S,T) \in H\}$ is equicontinuous. Then it follows from [16, Thm. 2.4] that, for any bounded set D in A and a neighborhood W of 0 in A, there exists a constant $r > 0$ such that $H \subseteq rN(D, W)$. Thus, H is u-bounded.

(b) By (a), the identity map $i : (M_d(A), s) \to (M_d(A), u)$ is bounded. So, by hypothesis, i is continuous and hence $u \le s$.

Now, let U be a proper topological algebra which contains A as a closed two-sided ideal. For any $y \in U$, define $L_y, R_y : A \to A$ by $L_y(a) = ya$ and $R_y(a) = ay$, $a \in A$. Clearly, $(L_y, R_y) \in M_d(A)$. Define a map $\mu' = U \to M_d(A)$ by $\mu'(y) = (L_y, R_y)$, $y \in U$. Then, as in [2, Prop. 3.7], μ' is a unique homomorphism satisfying $\mu'(a) = \mu(a)$ for all $a \in A$; further, $\operatorname{Ker}\mu' = \{y \in U : yA = \{0\}\}$. We next define a topology s' on U as the linear topology which has a base of neighborhoods of 0 in U consisting of all the sets of the form

$$N(D, W) = \{y \in U : yD \subseteq W, Dy \subseteq W\}, \tag{3.12}$$

where D is a finite subset of A and W is a neighborhood of 0 in A. If u' is the given topology on U, then, clearly, $s' \le u'$. □

Theorem 3.5. *Let U, A, μ', and s' be as above. Then*
(a) *$\mu' : (U, s') \to (M_d(A), s)$ is continuous and open onto $\mu'(U)$.*
(b) *Suppose that A is complete and metrizable, and has two-sided approximate identity. If (U, s') is complete and $\operatorname{Ker}\mu' = \{0\}$, then $\mu'(U) = M_d(A)$.*

Proof. (a) Let $\{y_\alpha\}$ be a net in U with $y_\alpha \xrightarrow{s'} y \in U$. This means, for each $a \in A$, $y_\alpha a \to ya$ and $ay_\alpha \to ay$ in A, that is $(L_{y_\alpha}, R_{y_\alpha}) \xrightarrow{s'} (L_y, R_y)$ in $M_d(A)$. Thus, μ' is continuous and open onto $\mu'(U)$.

(b) It follows from the hypothesis that $\mu'(U)$ is complete and hence a closed subset of $(M_d(A), s)$. Since $A \subseteq U$ and $\mu'(A)$ is s-dense in $M_d(A)$ (Theorem 3.2(b)), $\mu'(U)$ is s-dense in $M_d(A)$. Thus, $\mu'(U) = M_d(A)$. □

In the following paragraph, we denote the first and the second component of a double multiplier T on an algebra by $[T]'$ and $[T]''$, respectively. Let A and B be two proper Hausdorff topological algebras, and let $\phi : A \to B$ be an onto homomorphism.

Then by [7, Thm. 4] and [2, Prop. 3.8], there exists a unique (extension) homomorphism map $\tilde{\phi} : M_d(A) \to M_d(B)$ given by

$$[\tilde{\phi}(T)]'(\phi(a)) = \phi([T]'(a)), \qquad [\tilde{\phi}(T)]''(\phi(a)) = \phi([T]''(a)) \tag{3.13}$$

for $T \in M_d(A)$ and $a \in A$.

Theorem 3.6. *Let A, B, ϕ, and $\tilde{\phi}$ be as above, and let s_A and s_B denote the strong operator topologies on $M_d(A)$ and $M_d(B)$, respectively. If $\phi : A \to B$ is continuous, then so is the map $\tilde{\phi} : (M_d(A), s_A) \to (M_d(B), s_B)$.*

Proof. Suppose that $\phi : A \to B$ is continuous, and let $\{T_\alpha\}$ be a net in $M_d(A)$ with $T_\alpha \xrightarrow{S_A} T \in M_d(A)$. Then, if $b \in B$ and $b = \phi(a)$ for $a \in A$,

$$[\tilde{\phi}(T_\alpha)]'(b) = \phi([T_\alpha]'(a)) \longrightarrow \phi([T]'(a)) = [\tilde{\phi}(T)]'(b), \tag{3.14}$$

and similarly $[\tilde{\phi}(T_\alpha)]''(b) \to [\tilde{\phi}(T)]''(b)$ in B. Thus, $\tilde{\phi}(T_\alpha) \xrightarrow{S_B} \tilde{\phi}(T)$, showing that $\tilde{\phi}$ is continuous. □

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Khan: Department of Mathematics, King Abdul Aziz University, P.O. Box 9028, Jeddah 21413, Saudi Arabia

Mohammad: Department of Mathematics, Quaid-i-Azam University, Islamabad-45320, Pakistan

Thaheem: Department of Mathematical sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

E-mail address: athaheem@kfupm.edu.sa