# CONVEX AND STARLIKE CRITERIA

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ABSTRACT. We investigate an expression involving the quotient of the analytic representations of convex and starlike functions. Sufficient conditions are found for functions to be starlike of a positive order and convex.

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**1. Introduction.** Let *S* denote the class of functions *f* normalized by f(0) = f'(0) - 1 = 0 that are analytic and univalent in the unit disk  $\Delta = \{z : |z| < 1\}$ . A function *f* in *S* is said to be starlike of order  $\alpha$ ,  $0 \le \alpha < 1$ , and is denoted by  $S^*(\alpha)$  if  $\operatorname{Re}\{zf'(z)/f(z)\} > \alpha, z \in \Delta$ , and is said to be convex and is denoted by *K* if  $\operatorname{Re}\{1+zf''(z)/f'(z)\} > 0, z \in \Delta$ . Mocanu [9] studied linear combinations of the representations of convex and starlike functions and defined the class of  $\alpha$ -convex functions. In [8], it was shown that if

$$\operatorname{Re}\left[\alpha(1+zf''(z)/f'(z)) + (1-\alpha)zf'(z)/f(z)\right] > 0$$
(1.1)

for  $z \in \Delta$ , then *f* is starlike for  $\alpha$  real and convex for  $\alpha \ge 1$ .

In this note, we investigate the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In particular, we consider the class  $G_b$  consisting of normalized functions f defined by

$$G_{b} = \left\{ f : \left| \left( \frac{1 + z f''(z) / f'(z)}{z f'(z) / f(z)} \right) - 1 \right| < b, \ z \in \Delta \right\}.$$
 (1.2)

We determine sharp values of *b* for which  $G_b \,\subset\, S^*(\alpha), 1/2 \leq \alpha < 1$ , and also find values of *b* for which  $G_b \subset K$ . It is known ([7, 10]) that  $K \subset S^*(1/2)$ . We show that  $G_1 \subset S^*(1/2) - K$ . We also find values of *b* for which  $G_b$  is not starlike and not univalent.

We make use of the following lemma obtained by Jack in [4].

**LEMMA A.** Suppose  $\omega$  is analytic for  $|z| \le r$ ,  $\omega(0) = 0$  and  $|\omega(z_0)| = \max_{|z|=r} |\omega(z)|$ . Then  $z_0 \omega'(z_0) = k \omega(z_0)$ ,  $k \ge 1$ .

### 2. Main results

**THEOREM 1.** If  $0 < b \le 1$  and  $G_b$  is defined by (1.2), then  $G_b \subset S^*(2/(1+\sqrt{1+8b}))$ . The result is sharp for all b.

We prove this theorem in an equivalent form, which we write as

**THEOREM 1a.** Set  $b = (1 - \alpha)/2\alpha^2$ ,  $1/2 \le \alpha < 1$ . Then  $G_b \subset S^*(\alpha)$ , with extremal function  $z/(1-z)^{2(1-\alpha)}$ .

**PROOF OF THEOREM 1a.** It is well known that if  $\omega(z)$  is analytic in  $\Delta$  with  $\omega(0) = 0$ , then  $\operatorname{Re}\left(\frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}\right) > \alpha, z \in \Delta$ , if and only if  $\omega(z)$  is a Schwarz function, i.e.,  $|\omega(z)| < 1$  for  $z \in \Delta$  with  $\omega(0) = 0$ . Set

$$p(z) = \frac{zf'(z)}{f(z)} = \frac{1 + (1 - 2\alpha)\omega(z)}{1 - w(z)}$$
(2.1)

Then

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}$$
(2.2)

and

$$\left| \left( \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \right) - 1 \right| = \left| \frac{zp'(z)}{(p(z))^2} \right| = \left| \frac{2(1 - \alpha)z\omega'(z)}{(1 + (1 - 2\alpha)\omega(z))^2} \right|.$$
 (2.3)

If  $f \notin S^*(\alpha)$ , then by Lemma A there is a  $z_0 \in \Delta$  for which  $|\omega(z_0)| = 1$  and  $z_0 \omega'(z_0) \ge \omega(z_0)$ . It then follows from (2.3) that  $\left|\frac{z_0 p'(z_0)}{(p(z_0))^2}\right| \ge \frac{2(1-\alpha)}{(2\alpha)^2}$  which contradicts our hypothesis. This completes the proof.

**COROLLARY 1.**  $G_1 \subset S^*(1/2)$ .

**PROOF.** Set b = 1 in Theorem 1.

**COROLLARY 2.** If 
$$\operatorname{Re}\left(\frac{zf'(z)/f(z)}{1+zf''(z)/f'(z)}\right) > 1/2$$
 for  $z \in \Delta$ , then  $f \in S^*(1/2)$ 

**PROOF.** This follows from Corollary 1 upon noting that for any complex value w,  $|w-1| < 1 \iff \text{Re}(1/w) > 1/2$ .

We next give a partial converse to Corollary 1.

**THEOREM 2.** If  $f \in S^*(1/2)$ , then  $\left| \left( \frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} \right) - 1 \right| < 1$  for  $|z| < (2\sqrt{3}-3)^{1/2} = 0.68...$  The result is sharp.

**PROOF.** Set  $p(z) = zf'(z)/f(z) = 1/(1 - \omega(z))$ , where  $\omega(z)$  is a Schwarz function. We need to find the largest disk |z| < R for which  $|zp'(z)/p(z))^2| = |z\omega'(z)| < 1$ . Dieudonné [2] found the region of values for the derivative of Schwarz functions. This led to the sharp bound [3],

$$|\omega'(z)| \leq \begin{cases} 1, & r = |z| \leq \sqrt{2} - 1\\ \frac{(1+r^2)^2}{4r(1-r^2)}, & r \geq \sqrt{2} - 1. \end{cases}$$
(2.4)

Since  $|z\omega'(z)| \le (1+r^2)^2/4(1-r^2) = 1$  for  $r = (2\sqrt{3}-3)^{1/2}$ , the proof is complete.

**3.** A counterexample. The extreme points of the closed convex hull of convex functions and functions starlike of order 1/2 are identical. See [1]. Since  $G_1 \subset S^*(1/2)$ , one might, also, expect to have  $G_1 \subset K$ . Surprisingly, this is not the case. We now construct a function  $f \in G_1 - K$ .

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# **THEOREM 3.** $G_1 \notin K$ .

**PROOF.**  $G_1 \subset S^*(1/2)$ . Any of  $f \in G_1$  satisfies  $zf'(z)/f(z) = 1/(1-\omega(z))$  for some Schwarz function  $\omega(z)$ . Setting  $\alpha = 1/2$  in (2.3), we see that  $f \in G_1 \iff |z\omega'(z)| < 1$  for  $z \in \Delta$ , which means that  $z\omega'(z)$  must, also, be a Schwarz function. Since  $1 + zf''(z)/f(z) = (1 + z\omega'(z))/(1 - \omega(z))$ , it suffices to construct a Schwarz function  $\Omega(z) = z\omega'(z)$  for which

$$\operatorname{Re}\left\{\frac{1+\Omega(z)}{1-\omega(z)}\right\} < 0 \tag{3.1}$$

at some point  $z \in \overline{\Delta}$ . Let

$$A = \{ z \in \Delta : |z - z_0| < 10^{-5}, z_0 = e^{\pi i/4} = e^{i\theta_0} \},$$
(3.2)

and set

$$\phi(z) = (z_0 + \bar{z}_0) [(1 - \bar{z}_0 z)^{1/N} - 1], \qquad (3.3)$$

where *N* is large enough so that  $|\phi(z)/z| < 10^{-4}$  for  $z \in \Delta - A$  and  $|\operatorname{Im} \phi(z)| < 10^{-8}$  for  $z \in A$ . Define  $\Omega$  by  $\Omega(z) = 0.9999(z + \phi(z))$ .

We first show that  $\Omega(z)$  (and, consequently,  $\omega(z)$ ) is a Schwarz function and then show that inequality (3.1) holds when  $z = z_0$ .

If

$$z \in \Delta - A, \tag{3.4}$$

then

$$|\Omega(z)| \le 0.9999 (|z| + |\phi(z)|) \le 0.9999 (1.0001) < 1.$$
(3.5)

If  $z \in A$ , set  $z = z_0 - \epsilon e^{i\beta}$ ,  $0 < \epsilon < 10^{-5}$ , and note that  $-2\cos\theta_0 \le \operatorname{Re}\phi(z) \le 0$ . If  $\operatorname{Re}(z + \phi(z)) \ge 0$ , then  $|z + \operatorname{Re}\phi(z)| \le |z| < 1$ . If  $\operatorname{Re}(z + \phi(z)) < 0$ , then

$$|z + \operatorname{Re} \phi(z)| \le \sqrt{(\cos \theta_0 + \epsilon)^2 + (\sin \theta_0 + \epsilon)^2} < \sqrt{1 + 4\epsilon} < 1 + 2\epsilon < 1.0001.$$
(3.6)

Thus, if  $z \in A$ ,

$$|\Omega(z)| \le 0.9999 |z + \operatorname{Re} \phi(z)| + |\operatorname{Im} \phi(z)| < 0.9999 (1.0001) + 10^{-8} = 1.$$
(3.7)

Therefore,  $\Omega(z)$  is a Schwarz function.

We now show that (3.1) holds at  $z = z_0$  for this choice of  $\Omega(z)$ . Since

$$\left|\frac{\Omega(z)}{z} - 1\right| = |\omega'(z) - 1| < 0.0002 \quad \text{for } z \in \Delta - A,$$
(3.8)

we may write  $\omega(z) = z + \eta(z)$ , where  $|\eta(z)| < 0.0003$  for  $z \in A$ . Note that

$$(|1 - \omega(z_0)|^2) \operatorname{Re}\left(\frac{1 + \Omega(z_0)}{1 - \Omega(z_0)}\right) = \operatorname{Re}\left\{(1 - \Omega(z_0))(1 + \overline{\omega(z_0)})\right\}$$
  
=  $\operatorname{Re}\left\{(1 - 0.9999\bar{z}_0)(1 - \bar{z}_0 - \overline{\eta(z_0)})\right\}$   
 $\leq 1 - 1.9999\cos\theta_0 + 0.9999\cos 2\theta_0 + 2|\eta(z_0)|$   
 $< 1 - 1.9999\cos(\pi/4) + 0.0006 < 0.$  (3.9)

Hence, the function f for which  $1 + zf''(z)/f'(z) = (1 + \Omega(z))/(1 - \omega(z))$  must be in  $G_1 - K$ .

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**4. Convexity.** Since  $G_1 \notin K$ , we can ask if  $G_b \subset K$  for some b < 1. In general,  $S^*(\alpha) \notin K$  even for  $\alpha$  arbitrary close to 1 (*b* close to 0). To see this, we note that  $f_n(z) = z + a_n z^n$  is in  $S^*(\alpha)$  if and only if  $|a_n| \le (1-\alpha)/(n-\alpha)$  and  $f_n(z) \in K$  if and only if  $|a_n| \le 1/n^2$ . Thus,  $f(z) = z + (1-\alpha)/(n-\alpha) z^n \in S^*(\alpha) - K$  for  $n > 2/(1-\alpha)$ .

We next show that there are values of b for which the functions in  $G_b$  must be convex.

**THEOREM 4.**  $G_b \subset K$  for  $b \leq \sqrt{2}/2$ .

**PROOF.** Since  $f \in G_b \subset G_1 \subset S^*(1/2)$ , we may write  $zf'(z)/f(z) = 1/(1-\omega(z))$ , where  $\omega$  is a Schwarz function. For  $f \in G_b$ , we take  $\alpha = 1/2$  in (2.3) to obtain  $|z\omega'(z)| < \sqrt{2}/2$  and, consequently,  $|\omega(z)| < \sqrt{2}/2$ ,  $z \in \Delta$ . We need to show that

$$\operatorname{Re}\left\{1+zf''(z)/f'(z)\right\} = \operatorname{Re}\left\{\frac{(1+z\omega'(z))}{(1-\omega(z))}\right\} > 0.$$
(4.1)

Since

$$\left| \arg\left(\frac{1+z\omega'(z)}{1-\omega(z)}\right) \right| \le \left| \arg\left(1+z\omega'(z)\right) \right| + \left| \arg\left(1-\omega(z)\right) \right|$$
  
$$\le \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2},$$
(4.2)

the result follows.

In [6], MacGregor found the radius of convexity for  $S^*(1/2)$  to be  $(2\sqrt{3}-3)^{1/2} = 0.68...$  Since  $G_1 \subset S^*(1/2)$ , we know that the radius of convexity is at least this large. The following consequence of Theorem 4 is that functions in  $G_1$  are convex in the disk  $|z| < \sqrt{2}/2$ .

**COROLLARY.** If  $f \in G_b$ ,  $\sqrt{2}/2 \le b \le 1$ , then f is convex in the disk  $|z| < \sqrt{2}/2b$ .

**PROOF.** If  $|z\omega'(z)| < 1$  for  $z \in \Delta$ , then  $|z\omega'(z)| < t$  for |z| < t < 1. If  $f \in G_b$ , then  $|z\omega'(z)| < b$  for  $z \in \Delta$ . Hence,  $|z\omega'(z)| < \sqrt{2}/2$  when  $|z| < \sqrt{2}/2b$ .

**5. Examples.** Theorem 1 gives a sharp order of starlikeness for  $G_b$  when  $0 < b \le 1$ , with  $G_1 \subset S^*(1/2)$ . Our methods do not extend to b > 1, but we expect the order of starlikeness to decrease from 1/2 to 0 as b increases from 1 to some value  $b_0$  after which functions in  $G_b$  need not be starlike. We do not have a sharp result for b > 1, but our next example shows that the univalent functions in  $G_b$  are not necessarily starlike for  $b \ge 11.66$ .

The function  $h(z) = z(1-iz)^{i-1}$  is spiral-like [11] and, hence, in *S* because

$$\operatorname{Re}\left\{e^{\pi i/4}\frac{zh'(z)}{h(z)}\right\} = \frac{1}{\sqrt{2}}\left(\frac{1-|z|^2}{|1-iz|^2}\right) > 0, \quad z \in \Delta.$$
(5.1)

Since zh'(z)/h(z) = (1+z)/(1-iz), we see that h is not starlike for  $|z| < a, \sqrt{2}/2 < a < 1$ . Thus,  $f(z) = f_a(z) = h(az)/a$  is not starlike for  $z \in \Delta$ . Setting p(z) = zf'(z)/f(z) = (1+az)/(1-aiz), we have

$$\left|\frac{zp'(z)}{(p(z))^2}\right| = \left|\frac{(1+i)az}{(1+az)^2}\right| \le \frac{\sqrt{2}a}{(1-a)^2} < 11.66$$
(5.2)

for *a* sufficiently close to  $\sqrt{2}/2$ . Hence,  $f \in G_b - S^*(0)$  for b = 11.66.

Finally, we show that the functions in  $G_b$  need not be univalent. In [5], it is shown for  $h(z) = z(1-iz)^{i-1}$  that  $g(z) = \int_0^z h(t)/t \, dt = (1-iz)^i - 1$  is not in *S* because  $g(z_0) = g(-z_0)$  for  $z_0 = i(e^{2\pi} - 1)/(e^{2\pi} + 1), |z_0| = 0.996...$  We, thus, conclude that for  $f(z) = g(cz)/c, c = 0.997, f \in G_b - S$  for *b* sufficiently large.

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