SOME APPLICATIONS OF A DIFFERENTIAL SUBORDINATION

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ABSTRACT. A number of interesting criteria were given by earlier workers for a normalized analytic function to be in the familiar class $\mathcal{S}^*$ of starlike functions. The main object of the present paper is to extend and improve each of these earlier results. An application associated with an integral operator $\mathcal{F}_c$ ($c > -1$) is also considered.

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1. Introduction. Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}) \quad (1.1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $\mathcal{S}^*$ be the class of starlike functions in $\mathcal{U}$, defined by (cf., e.g., [2, 11])

$$\mathcal{S}^* := \left\{ f(z) \in \mathcal{A}(1) : \Re \left(\frac{zf'(z)}{f(z)} \right) > 0, (z \in \mathcal{U}) \right\}. \quad (1.2)$$

For analytic functions $g(z)$ and $h(z)$ with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$, ($z \in \mathcal{U}$), and $g(z) = h(w(z))$. We denote this subordination by $g(z) \prec h(z)$.

For a function $f(z)$ belonging to the class $\mathcal{A}(1)$, Bernardi [1] defined the integral operator $\mathcal{F}_c$ as follows:

$$(\mathcal{F}_c f)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (c > -1; z \in \mathcal{U}). \quad (1.3)$$

We note that $\mathcal{F}_c f \in \mathcal{A}(n)$ if $f \in \mathcal{A}(n)$. In particular, the operator $\mathcal{F}_1$ was studied earlier by Libera [3]. (Also, see Owa and Srivastava [8, p. 126 et seq.]).

R. Singh and S. Singh [10] proved that if $f(z) \in \mathcal{A}(1)$ and

$$\Re \{f'(z) + zf''(z)\} > -\frac{1}{4}, \quad (z \in \mathcal{U}), \quad (1.4)$$

then $f(z) \in \mathcal{S}^*$.

Recently, Yi and Ding [12] improved the above-mentioned result of R. Singh and S. Singh [10] by showing that if $f(z) \in \mathcal{A}(1)$ and

$$\Re \{f'(z) + zf''(z)\} > 1 - \frac{3}{4(1 - \log 2)^2 + 2} \approx -0.263, \quad (z \in \mathcal{U}), \quad (1.5)$$

then $f(z) \in \mathcal{F}^*$.

Furthermore, Nunokawa and Thomas [6] proved that if $f(z) \in \mathcal{A}(1)$ and

$$\Re\{f'(z)\} > -0.0175\dots, \quad (z \in \mathcal{U}), \quad (1.6)$$

then $\mathcal{F}_1 f \in \mathcal{F}^*$

In this paper, we extend and improve each of these earlier results in [6, 12] and also consider an interesting application associated with the integral operator $\mathcal{F}_c$.

2. Preliminary results. The following results are required in our investigation.

LEMMA 1 (Yi and Ding [12, Lem. 1]). *Suppose that the function $\phi : \mathbb{C}^2 \times \mathcal{U} \rightarrow \mathbb{C}$ satisfies the condition $\Re\{\phi(ix, y; z)\} \leq \delta$ for all real x and $y \leq -(1/2)(1+x^2)$ and all $z \in \mathcal{U}$. If $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in $\mathcal{U}$ and*

$$\Re\{\phi(p(z), zp'(z); z)\} > \delta, \quad (z \in \mathcal{U}), \quad (2.1)$$

then $\Re\{p(z)\} > 0$ in $\mathcal{U}$.

LEMMA 2 (Owa and Nunokawa [7, Thm. 1]). *Let $p(z)$ be analytic in $\mathcal{U}$ with*

$$p(0) = 1, \quad p'(0) = \dots = p^{(n-1)}(0) = 0. \quad (2.2)$$

If $p(z)$ satisfies the inequality

$$\Re\{p(z) + \alpha zp'(z)\} > \beta, \quad (z \in \mathcal{U}), \quad (2.3)$$

then

$$\Re\{p(z)\} > \beta + (1-\beta) \left\{ 2 \int_0^1 \frac{d\rho}{1 + \rho^{n\Re(\alpha)}} - 1 \right\}, \quad (z \in \mathcal{U}), \quad (2.4)$$

where $\alpha \neq 0$, $\Re(\alpha) \geq 0$, and $\beta < 1$.

LEMMA 3 (Owa and Nunokawa [7, Ex. 1]). *Let $\alpha > 0$ and $\beta < 1$. If $f(z) \in \mathcal{A}(n)$ satisfies the inequality*

$$\Re\{f'(z) + \alpha zf''(z)\} > \beta, \quad (z \in \mathcal{U}), \quad (2.5)$$

then

$$\Re\{f'(z)\} > \beta + (1-\beta) \{2\delta(n, \alpha) - 1\}, \quad (z \in \mathcal{U}), \quad (2.6)$$

where

$$\delta(n, \alpha) = \int_0^1 \frac{d\rho}{1 + \rho^{n\alpha}}. \quad (2.7)$$

Incidentally, the value of $\delta(n, \alpha)$ in (2.7) can be expressed as the Gauss hypergeometric function

$${}_2F_1\left(1, \frac{1}{n\alpha}; 1 + \frac{1}{n\alpha}; -1\right) \quad (2.8)$$

which may also be rewritten in terms of the difference of two Digamma (or ψ -) functions

$$\frac{1}{2n\alpha} \left[\psi\left(\frac{1+n\alpha}{2n\alpha}\right) - \psi\left(\frac{1}{2n\alpha}\right) \right] \quad \left(\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \right). \quad (2.9)$$

We also note that the inequality (2.5) is equivalent to the subordination given by

$$f'(z) + \alpha z f''(z) \prec \frac{1 + (1 - 2\beta)z}{1 - z}. \quad (2.10)$$

3. Main results. The following theorem is a generalization of the main result of Yi and Ding [12].

THEOREM. Let $\delta(n, \alpha)$ be as defined in Lemma 3 and let $\theta = 0.911621907$, $\alpha \geq 0.17418$, and

$$\alpha - \frac{(1 - \alpha)^2}{3\alpha} \tan^2 \theta < \frac{2\delta(n, \alpha) - 1}{\{1 - \delta(n, \alpha)\} \{2\delta(n, 1) - 1\}}. \quad (3.1)$$

If $f \in \mathcal{A}(n)$ satisfies the inequality

$$\begin{aligned} & \Re \{f'(z) + \alpha z f''(z)\} \\ & > 1 - \frac{\frac{2}{\alpha} + \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right)}{\frac{2}{\alpha} + 4\{1 - \delta(n, 1)\} \{1 - \delta(n, \alpha)\} \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right)}, \quad (z \in \mathcal{U}), \end{aligned} \quad (3.2)$$

then $f(z) \in \mathcal{P}^*$.

PROOF. Making use of Lemma 3 and the inequality (3.2), we obtain

$$\begin{aligned} & \Re \{f'(z)\} > \beta + (1 - \beta) \{2\delta(n, \alpha) - 1\} \\ & = 2\{\delta(n, \alpha) - 1\} \left[\frac{\frac{2}{\alpha} + \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right)}{\frac{2}{\alpha} + 4\{1 - \delta(n, 1)\} \{1 - \delta(n, \alpha)\} \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right)} \right] + 1 \\ & =: \gamma, \quad (z \in \mathcal{U}), \end{aligned} \quad (3.3)$$

where

$$\beta = 1 - \frac{\frac{2}{\alpha} + \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right)}{\frac{2}{\alpha} + 4\{1 - \delta(n, 1)\} \{1 - \delta(n, \alpha)\} \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right)}. \quad (3.4)$$

Since $\alpha \geq 0.17418$ and

$$\frac{1}{2} < \delta(n, \alpha) < 1, \quad (\alpha > 0; n \in \mathbb{N}), \quad (3.5)$$

we have

$$\frac{2}{\alpha} + 4\{1 - \delta(n, 1)\} \{1 - \delta(n, \alpha)\} \left(1 - \frac{(1 - \alpha)^2}{3\alpha^2} \tan^2 \theta\right) > 0. \quad (3.6)$$

Hence, by (3.1), we find from (3.3) that

$$0 < \gamma < 1. \quad (3.7)$$

If we put $p(z) = z^{-1}f(z)$, then

$$\Re \{f'(z)\} = \Re \{p(z) + zp'(z)\} > \gamma, \quad (z \in \mathcal{U}), \quad (3.8)$$

which, in view of Lemma 2, implies that

$$\Re \left\{ \frac{f(z)}{z} \right\} > \gamma + (1-\gamma) \{2\delta(n, 1) - 1\}, \quad (z \in \mathcal{U}). \quad (3.9)$$

By using (3.5) and (3.7), we get

$$\Re \left\{ \frac{f(z)}{z} \right\} > 0, \quad (z \in \mathcal{U}). \quad (3.10)$$

Next, we let

$$q(z) = \frac{zf'(z)}{f(z)} \quad \text{and} \quad \lambda(z) = \frac{f(z)}{z}. \quad (3.11)$$

Then

$$\Re \{\lambda(z)\} > \gamma + (1-\gamma) \{2\delta(n, 1) - 1\}, \quad (z \in \mathcal{U}) \quad (3.12)$$

and

$$\begin{aligned} f'(z) + \alpha z f''(z) &= \lambda(z) \left[\alpha z q'(z) + (1-\alpha)q(z) + \alpha \{q(z)\}^2 \right] \\ &= \phi(q(z), zq'(z); z), \end{aligned} \quad (3.13)$$

where $\phi(u, v; z) = \lambda(z)[\alpha u^2 + (1-\alpha)u + \alpha v]$.

By setting $\lambda(z) = a + bi$, we get

$$\begin{aligned} \Re \{\phi(ix, y; z)\} &\leq -\frac{1}{2} \{3\alpha ax^2 + 2b(1-\alpha)x + \alpha a\} \\ &\leq -\frac{a}{2} \left\{ \alpha - \frac{1}{3\alpha} (1-\alpha)^2 \left(\frac{b}{a} \right)^2 \right\} \end{aligned} \quad (3.14)$$

for all real x and $y \leq -(1/2)(1+x^2)$. Since $\Re \{f'(z)\} > 0$ ($z \in \mathcal{U}$) implies that $\lambda(z) < L(z) := -1 - (2/z) \log(1-z)$, we have $\lambda(\mathcal{U}) \subset L(\mathcal{U})$, where (see [9])

$$L(\mathcal{U}) \subset \{\omega : \Re(\omega) > 2 \log 2 - 1\} \cap \{\omega : |\Im(\omega)| < \pi\} \cap \{\omega : |\arg(\omega)| < \theta = 0.911621907\}. \quad (3.15)$$

By using (3.9) and (3.14), we obtain

$$\begin{aligned} \Re \{\phi(ix, y; z)\} &\leq -\frac{a}{2} \left\{ \alpha - \frac{(1-\alpha)^2}{3\alpha} \tan^2 \theta \right\} \\ &\leq \beta, \quad (z \in \mathcal{U}). \end{aligned} \quad (3.16)$$

Hence, by Lemma 1, we get

$$\Re \{q(z)\} = \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad (z \in \mathcal{U}). \quad (3.17)$$

This evidently completes the proof of the theorem. $\square$

COROLLARY 1. Let $\theta = 0.911621907$, $\alpha \geq 0.17418$, and

$$\alpha - \frac{(1-\alpha)^2}{3\alpha} \tan^2 \theta < \frac{2\delta(1, \alpha) - 1}{\{1 - \delta(1, \alpha)\}(2 \log 2 - 1)}. \quad (3.18)$$

If $f \in \mathcal{A}(1)$ satisfies the inequality

$$\Re\{f'(z) + \alpha z f''(z)\} > 1 - \frac{\frac{2}{\alpha} + \left(1 - \frac{(1-\alpha)^2}{3\alpha^2} \tan^2 \theta\right)}{\frac{2}{\alpha} + 4(1-\log 2)\{1 - \delta(1, \alpha)\} \left(1 - \frac{(1-\alpha)^2}{3\alpha^2} \tan^2 \theta\right)}, \quad (z \in \mathcal{U}), \quad (3.19)$$

then $f(z) \in \mathcal{P}^*$.

REMARK 1. For $\alpha = 1$, Corollary 1 immediately yields the main result of Yi and Ding [12, Thm., p. 614].

REMARK 2. A result of Ponnusamy [9, Thm. 4] can be obtained by taking $\beta = 0$ in the proof of our theorem.

It is not difficult to apply the definition (1.3) in order to show that

$$f'(z) = (\mathcal{F}_c f)'(z) + \frac{1}{c+1} z (\mathcal{F}_c f)''(z). \quad (3.20)$$

Thus, by the theorem, we arrive at the following application:

COROLLARY 2. Let $\theta = 0.911621907$, $-1 < c \leq 4.741187$, and

$$\frac{1}{c+1} - \frac{c^2}{3(c+1)} \tan^2 \theta < \frac{2\delta\left(n, \frac{1}{c+1}\right) - 1}{\left\{1 - \delta\left(1, \frac{1}{c+1}\right)\right\} \{2\delta(n, 1) - 1\}}. \quad (3.21)$$

If $f \in \mathcal{A}(n)$ satisfies the inequality

$$\Re\{f'(z)\} > 1 - \frac{2(c+1) + \left(1 - \frac{1}{3} c^2 \tan^2 \theta\right)}{2(c+1) + 4\{1 - \delta(n, 1)\} \left\{1 - \delta\left(n, \frac{1}{c+1}\right)\right\} \left(1 - \frac{1}{3} c^2 \tan^2 \theta\right)}, \quad (z \in \mathcal{U}) \quad (3.22)$$

then $\mathcal{F}_c f \in \mathcal{P}^*$, where $\mathcal{F}_c$ is defined by (1.3).

By setting $c = n = 1$ in Corollary 2, we obtain Corollary 3 below, which shows that the constant -0.0175 in the inequality (1.6) of Nunokawa and Thomas [6] can be reduced further.

COROLLARY 3. Let $\theta = 0.911621907$. If $f \in \mathcal{A}(1)$ satisfies the inequality

$$\Re\{f'(z)\} > 1 - \frac{5 - (1/3) \tan^2 \theta}{4 + 8(1 - \log 2)^2 (1 - (1/3) \tan^2 \theta)} \approx -0.025311\dots, \quad (z \in \mathcal{U}), \quad (3.23)$$

then $\mathcal{F}_1 f \in \mathcal{P}^*$.

PROOF. Since

$$\frac{1}{2} - \frac{1}{6} \tan^2 \theta = 0.222356 \quad (\theta = 0.911621907) \quad \text{and} \quad \frac{3 - 4 \log 2}{(2 \log 2 - 1)^2} = 1.523967\dots, \quad (3.24)$$

the proof of Corollary 3 is completed by setting $c = n = 1$ in Corollary 2. $\square$

REMARK 3. Several nonsharp results, obtained by various other authors (cf., e.g., [9]), correspond to the further special cases of Corollary 2 when $c = 0$ and $c = 1$.

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