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RECURSIVE FORMULAE FOR THE MULTIPLICATIVE PARTITION FUNCTION

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Abstract. For a positive integer n, let $f(n)$ be the number of essentially different ways of writing n as a product of factors greater than 1, where two factorizations of a positive integer are said to be essentially the same if they differ only in the order of the factors. This paper gives a recursive formula for the multiplicative partition function $f(n)$.

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A multi-partite number of order j is a j-dimensional vector, the components of which are nonnegative integers. A partition of $(n_1,n_2,\dots,n_j)$ is a solution of the vector equation

$$\sum_k (n_{1k},n_{2k},\dots,n_{jk}) = (n_1,n_2,\dots,n_j) \tag{1}$$

in multi-partition numbers other than $(0,0,\dots,0)$. Two partitions which differ only in the order of the multi-partite numbers are regarded as identical. We denote by $p(n_1,n_2,\dots,n_j)$ the number of different partitions of $(n_1,n_2,\dots,n_j)$. For example, $p(3) = 3$ since $3 = 2+1 = 1+1+1$ and $p(2,1) = 4$ since $(2,1) = (2,0)+(0,1) = (1,0)+(1,0)+(1,0) = (1,0)+(1,1)$. Let $f(1) = 1$ and for any integer $n > 1$, let $f(n)$ be the number of essentially different ways of writing n as a product of factors greater than 1, where two factorizations of a positive integer are said to be essentially the same if they differ only in the order of the factors. For example, $f(12)p(2,1) = 4$ since $12 = 2\cdot 6 = 3\cdot 4 = 2\cdot 2\cdot 3$. In general, if $n = p_1^{n_1}p_2^{n_2}\cdots p_j^{n_j}$, then $f(n) = p(n_1,n_2,\dots,n_j)$. We find recursive formulas for the multi-partite partition function $p(n_1,n_2,\dots,n_j)$. The most useful formula known to this day for actual evaluation of the multi-partite partition function is presented in Theorem 4.

For convenience, we define some sets used in this paper. For a positive integer r, let M_r^0 be the set of r-dimensional vectors with nonnegative integer components and M_r be the set of r-dimensional vectors with nonnegative integer components not all of which are zero. The following three theorems are well known.

Theorem 1 (Euler [3]; see also [1, p. 2]). *If $n \ge 0$, then*

$$p(n) = \sum_{m=1}^{\infty} (-1)^{m+1}\left(p\left(n-\frac{1}{2}m(3m-1)\right)+p\left(n-\frac{1}{2}(3m+1)\right)\right), \tag{2}$$

where we recall that $p(k) = 0$ for all negative integers k.

Theorem 2. *If $n \geq 0$, then $p(0) = 1$ and*

$$n \cdot p(n) = \sum_{k=1}^{n} \sigma(k) \cdot p(n-k), \tag{3}$$

where $\sigma(m) = \sum_{d|m} d$.

Theorem 3 ([1, Ch. 12]). *If $g(x_1, x_2, \ldots, x_r)$ is the generating function for $p(\vec{n})$ and $|x_i| < 1$ for $i \leq r$, then*

$$\begin{aligned} g(x_1, x_2, \ldots, x_r) &= \prod_{\vec{n} \in M_r} \frac{1}{1 - x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}} \\ &= 1 + \sum_{\vec{m} \in M_r} p(\vec{m}) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}. \end{aligned} \tag{4}$$

Similarly, we can extend the equation of Theorem 2 to multi-partite numbers as follows.

Theorem 4. *For $\vec{n} \in M_r$, we have*

$$n_i \cdot p(\vec{n}) = \sum_{\substack{l_j \leq n_j \text{ for } j \leq r \\ \vec{l} \in M_r}} \frac{\sigma(\gcd[\vec{l}])}{\gcd[\vec{l}]} \cdot l_i \cdot p(\vec{n} - \vec{l}). \tag{5}$$

Proof. Let $g(x_1, x_2, \ldots, x_r)$ be the function defined in Theorem 3. Taking the ith partial logarithmic derivative of the product formula for $g(x_1, x_2, \ldots, x_r)$ in (4), we get

$$\begin{aligned} \frac{\partial g(x_1, x_2, \ldots, x_r)}{\partial x_i} \cdot \frac{x_i}{g(x_1, x_2, \ldots, x_r)} &= \sum_{\vec{l} \in M_r} \frac{l_i \cdot \prod_{j=1}^{r} x_j^{l_j}}{1 - \prod_{j=1}^{r} x_j^{l_j}} \\ &= \sum_{\vec{l} \in M_r} \sum_{k=1}^{\infty} l_i \cdot \left(\prod_{j=1}^{r} x_j^{l_j} \right)^k. \end{aligned} \tag{6}$$

Taking the ith partial derivative of the right-hand side of (4), we get

$$\begin{aligned} \sum_{\vec{n} \in M_r} n_i \cdot p(\vec{n}) x_1^{n_1} x_2^{n_2} \cdot x_r^{n_r} &= \frac{\partial g(x_1, x_2, \ldots, x_r)}{\partial x_i} \cdot x_i \\ &= g(x_1, x_2, \ldots, x_r) \sum_{\vec{t} \in M_r} \sum_{k=1}^{\infty} t_i \cdot \left(\prod_{j=1}^{r} x_j^{t_j} \right)^k \\ &= \left(\sum_{\vec{m} \in M_r^0} p(\vec{m}) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} \right) \sum_{\vec{t} \in M_r} \sum_{k=1}^{\infty} t_i \cdot \left(\prod_{j=1}^{r} x_j^{t_j} \right)^k. \end{aligned} \tag{7}$$

Comparing the coefficients of both sides of (7), we get

$$
\begin{aligned}
n_i \cdot p(\vec{n}) &= \sum_{\substack{\vec{m},\vec{t}\in M_r^0,k\in M_1 \\ \vec{m}+k\vec{t}=\vec{n}}} t_i \cdot p(\vec{m}) \\
&= \sum_{\vec{l}\in M_r} p(\vec{n}-\vec{l}) \sum_{k|\gcd(\vec{l})} \frac{l_i}{k} \\
&= \sum_{\substack{l_j\le n_j \text{ for } j\le r \\ \vec{l}\in M_r}} \frac{\sigma(\gcd[\vec{l}])}{\gcd[\vec{l}]} \cdot l_i \cdot p(\vec{n}-\vec{l}).
\end{aligned}
\tag{8}
$$

The theorem is proved. □

Corollary 5. *For $\vec{n} \in M_r$, we have*

$$
\left(\sum_{i=1}^{r} n_i\right) \cdot p(\vec{n}) = \sum_{\substack{l_j\le n_j \text{ for } j\le r \\ \vec{l}\in M_r}} \frac{\sigma(\gcd[\vec{l}])}{\gcd[\vec{l}]} \left(\sum_{i=1}^{r} l_i\right) \cdot p(\vec{n}-\vec{l}). \tag{9}
$$

For positive integers m and n, let

$$
(m,n)_{\vDash} = \max_{\substack{k|m \\ n^{1/k} \text{ is an integer}}} k. \tag{10}
$$

The following properties of $(m,n)_{\vDash}$ are easy to obtain:

(1) $(m, p_1^{n_1} p_2^{n_2} \cdot p_k^{n_k}) = \gcd(m, n_1, n_2, \dots, n_k)$

(2) $(m, nk)_{\vDash} = \gcd[(m,n)_{\vDash}, (mk)_{\vDash}]$ *for* $\gcd(n,k) = 1$

(3) $(mk, n)_{\vDash} = (m,n)_{\vDash} \cdot (k,n)_{\vDash}$ *for* $\gcd(m,k) = 1$.

From the point of view of the multiplicative partition function, Theorem 4 can be restated as the following theorem.

Theorem 6. *let n, t be positive integers and let p be a prime number such that $p \nmid m$. Then we get*

$$
t \cdot f(mp^t) = \sum_{i=1}^{t} \sum_{l|m} \frac{\sigma((i,l)_{\vDash})}{(i,l)_{\vDash}} \, i \cdot f\left(\frac{m}{l} p^{t-i}\right). \tag{11}
$$

In [4], MacMahon presents a table of values of $f(n)$ for those n which divide one of $2^{10} \cdot 3^8$, $2^{10} \cdot 3 \cdot 5$, $2^9 \cdot 3^2 \cdot 5^1$, $2^8 \cdot 3^3 \cdot 5^1$, $2^6 \cdot 3^2 \cdot 5^2$, $2^5 \cdot 3^3 \cdot 5^2$. In [2], Canfield, Erdös, and Pomerance commented that they doubted the correctness of MacMahon's figures. Specifically,

$$
\begin{aligned}
p(10,5) &= 3804, &&\text{not } 3737, &(12)\\
p(9,8) &= 13715, &&\text{not } 13748, &(13)\\
p(10,8) &= 21893, &&\text{not } 21938, &(14)\\
p(4,1,1) &= 38, &&\text{not } 28. &(15)
\end{aligned}
$$

From Theorem 4 we can easily be sure that Canfield, Erdös and Pomerance comment is true.

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