

## ON AZUMAYA GALOIS EXTENSIONS AND SKEW GROUP RINGS

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**ABSTRACT.** Two characterizations of an Azumaya Galois extension of a ring are given in terms of the Azumaya skew group ring of the Galois group over the extension and a Galois extension of a ring with a special Galois system is determined by the trace of the Galois group.

**Keywords and phrases.** Azumaya algebras, Galois extensions,  $H$ -separable extensions, skew group rings.

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**1. Introduction.** Let  $S$  be a ring with 1,  $G$  a finite automorphism group of  $S$  of order  $n$  for some integer  $n$  invertible in  $S$ ,  $S^G$  the subring of the elements fixed under each element in  $G$ ,  $C$  the center of  $S$ , and  $S^*G$  the skew group ring of  $G$  over  $S$ . In [3] and [2],  $S$  is called an Azumaya Galois extension of  $S^G$  if it is a  $G$ -Galois extension of  $S^G$  which is an Azumaya  $C^G$ -algebra. It was shown that  $S$  is an Azumaya Galois extension if and only if  $S^*G$  is an Azumaya  $C^G$ -algebra. The purpose of the present paper is to give two more characterizations of an Azumaya Galois extension in terms of the Azumaya skew group ring  $S^*G$ . We show that  $S$  is an Azumaya  $G$ -Galois extension if and only if  $S^*G$  is an Azumaya algebra over its center  $Z$ , a  $G'$ -Galois extension with an inner Galois group  $G'$  induced by the elements of  $G$ , and  $ZG$  is a finitely generated projective  $C^G$ -module of rank  $n$ . Moreover, for the skew group ring  $S^*G$ , where  $S$  is a separable  $C^G$ -algebra, an expression of the commutator subring of  $C$  in  $S^*G$  is obtained by using  $S$  and its commutator subring in  $S^*G$ . Furthermore, let  $H$  be a normal subgroup of  $G$ ,  $K$  the commutator subgroup of  $H$  in  $G$ , and  $H'$  the inner automorphism group of  $S^*G$  induced by the elements of  $H$  ( $K'$  and  $(G/H)'$  are similarly defined). Then, it is shown that  $(S^*G)^{K'}$  is a  $(G/K)'$ -Galois extension with a Galois system  $\{m^{-1}g_j, g_j^{-1}/g_j \text{ in } H\}$  if and only if  $\text{Tr}_{G'}(g_i) = 0$  for each  $g_i$  not in  $K$ , where  $m$  is the order of  $H$  for some integer  $m$  and  $\text{Tr}_{G'}(g_i)$  is the trace of  $G'$  at  $g_i$ .

**2. Preliminaries.** Throughout, let  $S$  be a ring with 1,  $G = \{g_1, \dots, g_n\}$  for some integer  $n$  invertible in  $S$ ,  $C$  the center of  $S$ ,  $S^G$  the subring of the elements fixed under each element in  $G$ , and  $S^*G$  the skew group ring of  $G$  over  $S$ . Let  $B$  be a subring of a ring  $A$ . We call  $A$  a separable extension of  $B$  if there exist  $\{a_i, b_i\}$  in  $A$ ,  $i = 1, \dots, m$  for some integer  $m$ , such that  $\sum a_i b_i = 1$  and  $\sum a a_i \otimes b_i = \sum a_i \otimes b_i a$  for all  $a$  in  $A$ , where  $\otimes$  is over  $B$  and  $\{a_i, b_i\}$  is called a separable system for  $A$ . An Azumaya algebra is a separable extension over its center. A ring  $A$  is called an  $H$ -separable extension of  $B$  if  $A \otimes A$  is a direct summand of a finite direct sum of  $A$  as an  $A$ -bimodule, where  $\otimes$  over  $B$ . Denote the commutator subring of  $B$  in  $A$  by

$V_A(B)$ . An  $H$ -separable extension  $A$  over  $B$  is equivalent to the existence of an  $H$ -separable system  $\{d_i \text{ in } V_A(B); \sum(x_{ij} \otimes y_{ij}) \text{ in } V_{A \otimes A}(A)\}$ ,  $j = 1, \dots, u$  and  $i = 1, \dots, v$  for some integers  $u$  and  $v$  such that  $\sum d_i(\sum(x_{ij} \otimes y_{ij})) = 1 \otimes 1$ ,  $i = 1, \dots, v$  and  $j = 1, \dots, u$ . The ring  $S$  is called a  $G$ -Galois extension of  $S^G$  if there exist  $\{c_i, d_i \text{ in } S, i = 1, \dots, k \text{ for some integer } k\}$  such that  $\sum c_i d_i = 1$  and  $\sum a_i g_j(b_i) = 0$  for each  $g_j \neq 1$ , where  $\{c_i, d_i\}$  is called a  $G$ -Galois system for  $S$ . It is well known that an Azumaya algebra is an  $H$ -separable extension and that an  $H$ -separable extension is a separable extension. A skew group ring  $S^*G$  is a ring with a free basis  $\{g_i\}$  over  $S$  such that  $g_i s = (g_i(s))g_i$  for each  $g_i$  in  $G$  and  $s$  in  $S$ . We denote the center of  $S^*G$  by  $Z$ , the inner automorphism group of  $S^*G$  induced by the elements of the subgroup  $H$  of  $G$  by  $H'$  ( $= \{g' / g'(x) = gxg^{-1} \text{ for } g \text{ in } H \text{ and all } x \text{ in } S^*G\}$ ), and the commutator subgroup of  $H$  in  $G$  by  $V_G(H)$ .

**3. Skew group rings.** In this section, keeping the notations of Section 2, we give two characterizations of an Azumaya Galois extension and an expression of the commutator subring of  $C$  in  $S^*G$  when  $S$  is a separable  $C^G$ -algebra.

**THEOREM 3.1.** *The following statements are equivalent:*

- (i)  $S$  is an Azumaya Galois extension,
- (ii)  $S^*G$  is an Azumaya  $Z$ -algebra and  $S$  satisfies the double centralizer property in  $S^*G$ , and
- (iii)  $S^*G$  is an Azumaya  $Z$ -algebra and a  $G'$ -Galois extension of  $(S^*G)^{G'}$ , and  $ZG$  is a finitely generated and projective  $C^G$ -module of rank  $n$ .

**PROOF.** (i) $\Rightarrow$ (ii). Since  $S$  is an Azumaya Galois extension,  $S^*G$  is an Azumaya  $C^G$ -algebra (that is,  $Z = C^G$ ) and  $S^*G$  is an  $H$ -separable extension of  $S$  [3, Thm. 3.1]. Noting that  $S$  is a direct summand of  $S^*G$  as a left  $S$ -module, we conclude that  $V_{S^*G}(V_{S^*G}(S)) = S$  [6, Prop. 1.2].

(ii) $\Rightarrow$ (i). Since  $V_{S^*G}(V_{S^*G}(S)) = S$ ,  $Z$  is contained in  $S$ ; and so  $Z$  is contained in  $C$ . But then  $Z = C^G$ . This implies that  $S^*G$  is an Azumaya  $C^G$ -algebra by (ii). Thus,  $S$  is an Azumaya Galois extension [3, Thm. 3.1].

(i) $\Rightarrow$ (iii). Since the restriction of  $G'$  to  $S$  is  $G$ ,  $S^*G$  is a  $G'$ -Galois extension of  $(S^*G)^{G'}$  with the same Galois system as  $S$  (for  $S$  is  $G$ -Galois). Also, by hypothesis,  $S$  is an Azumaya Galois extension, so  $S^*G$  is an Azumaya  $C^G$ -algebra [3, Thm. 3.1]. Moreover, since  $Z = C^G$ ,  $ZG$  is a free  $Z$ -module of rank  $n$ .

(iii) $\Rightarrow$ (i). Since  $S^*G$  is a  $G'$ -Galois extension of  $(S^*G)^{G'}$  with an inner Galois group  $G'$ , it is an  $H$ -separable extension of  $(S^*G)^{G'}$  [7, Cor. 3]. But  $n$  is a unit in  $S$ , so  $V_{S^*G}((S^*G)^{G'})$  is a separable  $Z$ -algebra and a finitely generated and projective  $Z$ -module of rank  $n$  [7, Prop. 4]. Moreover,  $S^*G$  is a  $G'$ -Galois extension of  $(S^*G)^{G'}$ , so it is finitely generated and projective  $(S^*G)^{G'}$ -module. Since  $n$  is a unit in  $S$ ,  $ZG$  is a separable  $Z$ -algebra. But then  $V_{S^*G}((S^*G)^{G'}) = V_{S^*G}(V_{S^*G}(ZG)) = ZG$  by the commutator theorem for Azumaya algebras [4, Thm. 4.3]. Therefore,  $ZG$  is a finitely generated and projective  $Z$ -module of rank  $n$  [1, Prop. 4]. From the fact that there are  $n$  elements  $\{g_i\}$  of  $G$  as generators of  $ZG$ , it is not difficult to show that  $\{g_i\}$  are free over  $Z$ . Hence,  $Z$  is a finitely generated and projective  $C^G$ -module. Thus, the rank of  $ZG$  over  $C^G$  is a product of the rank of  $ZG$  over  $Z$  and the rank of  $Z$  over  $C^G$ ; that is,

$n = n(\text{rank of } Z \text{ over } C^G)$ . This implies that  $Z = C^G$ . Therefore,  $S^*G$  is an Azumaya  $C^G$ -algebra; and so  $S$  is an Azumaya Galois extension [3, Thm. 3.1].  $\square$

**COROLLARY 3.2.** *Let  $S$  be a separable  $C^G$ -algebra. If  $V_{S^*G}(S)$  is a  $G''$ -Galois extension, where  $G''$  is the inner automorphism group of  $V_{S^*G}(S)$  induced by and isomorphic with  $G$ , then  $S$  is an Azumaya Galois algebra.*

**PROOF.** Since  $V_{S^*G}(S)$  is a  $G''$ -Galois extension, there exists a  $G''$ -Galois system  $\{c_i, d_i \text{ in } V_{S^*G}(S) \mid i = 1, \dots, k\}$  for  $V_{S^*G}(S)$ . Then, it is straightforward to check that  $\{c_j; \sum g_j d_i \otimes g_j^{-1}, i = 1, \dots, k \text{ and } j = 1, \dots, m \text{ for some integers } k \text{ and } m\}$  is an  $H$ -separable system for  $S^*G$  over  $S$  [1, Thm. 1]. Hence,  $S$  satisfies the double centralizer property in  $S^*G$  [7, Prop. 1.2]. Moreover,  $n$  is a unit in  $S$ , so  $S^*G$  is a separable extension of  $S$ . By hypothesis,  $S$  is a separable  $C^G$ -algebra, so  $S^*G$  is a separable  $C^G$ -algebra by the transitivity of separable extensions. But then  $S^*G$  is an Azumaya  $Z$ -algebra. Therefore,  $S$  is an Azumaya Galois extension by Theorem 3.1.  $\square$

For the skew group ring  $S^*G$  of  $G$  over a separable  $C^G$ -algebra  $S$ , we next give an expression of  $V_{S^*G}(C)$  in terms of  $S$  and  $V_{S^*G}(S)$  (for more about  $V_{S^*G}(S)$ , see [1]).

**THEOREM 3.3.** *If  $S$  is a separable  $C^G$ -algebra, then*

- (i)  $CZ$  is a commutative separable subalgebra of  $S^*G$  and
- (ii)  $SZ, V_{S^*G}(S)$ , and  $V_{S^*G}(C)$  are Azumaya  $CZ$ -algebras contained in  $S^*G$ , such that  $V_{S^*G}(C) \cong SZ \otimes V_{S^*G}(S)$ , where  $\otimes$  is over  $CZ$ .

**PROOF.** (i) Since  $S$  is a separable  $C^G$ -algebra,  $C$  is also a separable  $C^G$ -algebra. Hence,  $C \otimes Z$  is a separable  $Z$ -algebra, where  $\otimes$  is over  $C^G$ ; and so the homomorphic image  $CZ$  of  $C \otimes Z$  is also a separable  $Z$ -algebra. Clearly,  $CZ$  is commutative.

(ii) Since  $n$  is a unit in  $S$ ,  $S^*G$  is a separable  $S$ -extension. Hence,  $S^*G$  is a separable  $C^G$ -algebra by the transitivity of separable extensions; and so  $S^*G$  is an Azumaya  $Z$ -algebra. But then  $V_{S^*G}(CZ)$  is a separable subalgebra of  $S^*G$  such that  $V_{S^*G}(V_{S^*G}(CZ)) = CZ$  [4, Thm. 4.3] (for  $CZ$  is a separable subalgebra of  $S^*G$  by (i)). This implies that the center of  $V_{S^*G}(CZ)$  is  $CZ$ . Thus,  $V_{S^*G}(CZ)$  is an Azumaya  $CZ$ -algebra. By hypothesis again,  $S$  is a separable  $C^G$ -algebra, so it is an Azumaya  $C$ -algebra. Hence,  $S \otimes CZ$  is an Azumaya  $CZ$ -algebra, where  $\otimes$  is over  $C$ . Thus,  $SZ$  is also an Azumaya  $CZ$ -algebra. Noting that  $SZ \subset V_{S^*G}(CZ)$ , we conclude that  $V_{S^*G}(CZ) \cong SZ \otimes V_{S^*G}(SZ)$ , where  $\otimes$  is over  $CZ$  [7, Thm. 4.3]. Moreover, since  $V_{S^*G}(CZ) = V_{S^*G}(C)$  and  $V_{S^*G}(SZ) = V_{S^*G}(S)$ , we conclude that  $V_{S^*G}(C) \cong SZ \otimes V_{S^*G}(S)$ , where  $\otimes$  is over  $CZ$ .  $\square$

By [3, Thm. 3.1], if  $S$  is an Azumaya Galois extension, then  $S^*G$  is an Azumaya  $C^G$ -algebra (that is,  $Z = C^G$ ) and  $S$  is a separable  $C^G$ -algebra. Thus, we have the following result.

**COROLLARY 3.4.** *If  $S$  is an Azumaya Galois extension, then  $V_{S^*G}(C) \cong S \otimes V_{S^*G}(S)$  as Azumaya  $C$ -algebras, where  $\otimes$  is over  $C$  such that  $V_{S^*G}(C)$  is a  $G'$ -Galois extension of  $V_{S^*G}(CG)$ .*

**PROOF.** By the above remark, it suffices to show that  $V_{S^*G}(C)$  is a  $G'$ -Galois extension of  $V_{S^*G}(CG)$ . In fact, since  $S$  is a  $G$ -Galois extension and  $S \subset V_{S^*G}(C)$ ,  $V_{S^*G}(C)$  is a  $G'$ -Galois extension with the same Galois system as  $S$  by noting that  $V_{S^*G}(C)$  is

$G'$ -invariant (for  $G$  is the restriction of  $G'$  to  $S$ ). Moreover, it is clear that  $(V_{S^*G}(C))^{G'} = V_{S^*G}(CG)$ .  $\square$

**4. A Galois system.** It is well known that  $\{n^{-1}g_i, g_i^{-1} \mid g_i \text{ in } G\}$  is a separable system for a separable group ring  $RG$  over a ring  $R$  with 1, where  $G = \{g_i \mid i = 1, \dots, n\}$  for some integer  $n$  invertible in  $R$ , for a separable skew group ring  $S^*G$  over  $S$  and for a separable projective group ring  $RG_f$  over  $R$  as defined in [9]. In this section, we give an equivalent condition for  $(S^*G)^{K'}$  to have a  $(G/K)$ -Galois system similar to the above separable system for a normal subgroup  $K$  of  $G$ .

**THEOREM 4.1.** *Let  $H$  be a normal subgroup of  $G$  and  $V_G(H) = K$ . Then*

- (i)  $K$  is a normal subgroup of  $G$  and
- (ii)  $\text{Tr}_{H'}(g_i) = 0$  for each  $g_i$  not in  $K$  if and only if  $(S^*G)^{K'}$  is a  $(G/K)$ -Galois extension of  $(S^*G)^{G'}$  with a Galois system  $\{m^{-1}g_j, g_j^{-1} \mid g_j \text{ in } H\}$ , where  $m$  is the order of  $H$ .

**PROOF.** (i) We want to show that  $g_i K g_i^{-1} \subset K$  for each  $g_i$  in  $G$ . For any  $x$  in  $K$  and  $y$  in  $H$ ,  $g_i x g_i^{-1} y = g_i x g_i^{-1} y g_i g_i^{-1} = g_i x z g_i^{-1}$ , where  $z = g_i^{-1} y g_i$ . Since  $H$  is normal in  $G$ ,  $z$  is in  $H$ . Hence,  $xz = zx$ . But then  $g_i x g_i^{-1} y = g_i x z g_i^{-1} = g_i z x g_i^{-1} = g_i g_i^{-1} y g_i x g_i^{-1} = y g_i x g_i^{-1}$ . This implies that  $g_i x g_i^{-1}$  is in  $K$ . Thus,  $K$  is normal in  $G$ .

(ii) Assume that  $\text{Tr}_{H'}(g_i) = 0$  for each  $g_i$  not in  $K$ . Then  $\sum g_j g_i g_j^{-1} = 0$ , where  $H = \{g_j \mid j = 1, \dots, m \text{ for some integer } m\}$ ; that is,  $\sum g_j g_i g_j^{-1} g_i^{-1} g_i = \sum g_j ((g_i)'(g_j^{-1})) g_i = 0$ ,  $j = 1, \dots, m$ . Hence,  $(m^{-1}) \sum g_j ((g_i)'(g_j^{-1})) = 0$  for each  $g_i$  not in  $K$ . Clearly, for each  $g_i$  in  $K$ ,  $(m^{-1}) \sum g_j ((g_i)'(g_j^{-1})) = 1$ . Thus,  $\{m^{-1}g_j, g_j^{-1} \mid g_j \text{ in } H\}$  is a  $(G/K)$ -Galois system for  $(S^*G)^{K'}$  (for  $H \subset (S^*G)^{K'}$ ), where  $m$  is the order of  $H$ .

Conversely,  $(m^{-1}) \sum g_j ((g_i)'(g_j^{-1})) = 0$  for each  $g_i$  not in  $K$ , so  $\sum g_j g_i g_j^{-1} g_i^{-1} = 0$ . Hence,  $\sum g_j g_i g_j^{-1} = 0$ ; that is,  $\text{Tr}_{H'}(g_i) = 0$  for each  $g_i$  not in  $K$ .  $\square$

We derive the following corollaries.

**COROLLARY 4.2.**  *$S^*G$  has a  $(G/K)$ -Galois system  $\{n^{-1}g_i, g_i^{-1} \mid g_i \text{ in } G\}$ , where  $K$  is the center of  $G$ , if and only if  $\text{Tr}_{G'}(g_i) = 0$  for each  $g_i$  not in  $K$ .*

**PROOF.** Let  $H$  be  $G$ . Then  $K =$  the center of  $G$ ; and so the corollary follows immediately from the theorem.  $\square$

**COROLLARY 4.3.**  *$S^*G$  has a  $G'$ -Galois system  $\{n^{-1}g_i, g_i^{-1} \mid g_i \text{ in } G\}$  if and only if  $\text{Tr}_{G'}(g_i) = 0$  for each  $g_i \neq 1$ .*

**PROOF.** This is the case of the theorem that the center of  $G$  is trivial.  $\square$

We derive an equivalent condition for a Galois subring of  $S^*G$  arising from a  $G'$ -invariant subring.

**COROLLARY 4.4.** *Let  $A$  be a  $G'$ -invariant subring of  $S^*G$  and  $H = \{g_i \text{ in } G \mid g_i(a) = a \text{ for each } a \text{ in } A\}$ . Then*

- (i)  $H$  is normal in  $G$  and
- (ii) Denoting  $(S^*G)^{H'}$  by  $B$  and  $V_G(H)$  by  $K$ ,  $\text{Tr}_{K'}(g_i) = 0$  for each  $g_i$  not in  $H$  if and only if  $\{m^{-1}g_j, g_j^{-1} \mid g_j \text{ in } K\}$  is a  $(G/H)$ -Galois system for  $B$ , where  $m$  is the

order of  $K$ .

**PROOF.** Part (i) is straightforward and part (ii) follows immediately from Theorem 4.1.  $\square$

We conclude the present paper with an example of an Azumaya skew group ring  $S^*G$  which is a  $G'$ -Galois extension such that the rank of  $ZG$  over  $C^G$  is not  $n$  (see Theorem 3.1(iii)). Hence,  $S$  is not an Azumaya Galois extension by Theorem 3.1.

Let  $R$  be the real field,  $S = R[i, j, k]$  the quaternion algebra over  $R$ , and  $G = \{1, g \mid g(x) = ix(i)^{-1} \text{ for each } x \text{ in } S\}$ . Then

(1)  $S$  is a  $G$ -Galois extension with a Galois system  $\{2^{-1}, 2^{-1}j; 1, -j\}$ . Hence,  $S^*G$  is a  $G'$ -Galois extension with the same Galois system.

(2) Since  $S^*G$  is a separable extension of  $S$  and  $S$  is an Azumaya  $R$ -algebra,  $S^*G$  is a separable  $R$ -algebra. Hence,  $S^*G$  is an Azumaya  $Z$ -algebra.

(3) The center  $Z$  of  $S^*G$  is  $(R + Ri)$  by direct computation.

(4)  $ZG$  is free over  $Z$  by direct verification.

(5)  $C = R$  and  $C^G = C = R$ .

(6)  $Z$  is a free  $R$ -module of rank 2 and  $ZG$  is a free  $C^G$ -module of rank 4 ( $\neq 2 =$  the order of  $G$ ), so one of the three conditions in Theorem 3.1(iii) does not hold.

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#### REFERENCES

- [1] R. Alfaro and G. Szeto, *The centralizer on  $H$ -separable skew group rings*, Rings, extensions, and cohomology (New York), Lecture Notes in Pure and Appl. Math., vol. 159, Marcel Dekker, Inc., 1994, Basal Hong Kong, pp. 1-7. MR 95g:16027. Zbl 812.16038.
- [2] ———, *Skew group rings which are Azumaya*, Comm. Algebra **23** (1995), no. 6, 2255-2261. MR 96b:16027. Zbl 828.16030.
- [3] ———, *On Galois extensions of an Azumaya algebra*, Comm. Algebra **25** (1997), no. 6, 1873-1882. CMP 97 11. Zbl 890.16017.
- [4] F. DeMeyer and E. Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Mathematics, vol. 181, Springer-Verlag, New York, 1971. MR 43#6199. Zbl 215.36602.
- [5] F. R. DeMeyer, *Some notes on the general Galois theory of rings*, Osaka J. Math. **2** (1965), 117-127. MR 32#128. Zbl 143.05602.
- [6] K. Sugano, *Note on semisimple extensions and separable extensions*, Osaka J. Math. **4** (1967), 265-270. MR 37#1412. Zbl 199.07901.
- [7] ———, *On a special type of Galois extensions*, Hokkaido Math. J. **9** (1980), no. 2, 123-128. MR 82c:16036. Zbl 467.16005.
- [8] G. Szeto, *Separable Subalgebras of a Class of Azumaya Algebras*, to appear in Internat. J. Math. Math. Sci.
- [9] G. Szeto and Y. F. Wong, *On Azumaya projective group rings*, Azumaya algebras, actions, and modules, Contemp. Math., vol. 124, Amer. Math. Soc., 1992, pp. 251-256. MR 92m:16024. Zbl 749.16013.

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