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FANTASTIC FILTERS OF LATTICE IMPLICATION ALGEBRAS

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ABSTRACT. The notion of a fantastic filter in a lattice implication algebra is introduced, and the relations among filter, positive implicative filter, and fantastic filter are given. We investigate an equivalent condition for a filter to be fantastic, and state an extension property for fantastic filter.

Keywords and phrases. Lattice implication algebra, filter, implicative filter, positive implicative filter, fantastic filter.

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1. Introduction. In order to research the logical system whose propositional value is given in a lattice, Xu [5] proposed the concept of lattice implication algebras, and discussed their some properties. Also, in [6], Xu and Qin discussed the properties of lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. For the general development of lattice implication algebras, the filter theory plays an important role as well as ideal theory. Xu and Qin [7] introduced the notion of filters in a lattice implication algebra, and investigated their properties. In [2], we gave an equivalent condition of a filter, and provided some equivalent conditions that a filter is an implicative filter, and using this result an extension property for implicative filter is constructed. Jun et al. [4] introduced the concepts of a positive implicative filter and an associative filter in a lattice H implication algebra. They proved that (i) every positive implicative filter is an implicative filter, and (ii) every associative filter is a filter. They also provided equivalent conditions for both a positive implicative filter and an associative filter. In [3], Jun et al. defined an LI-ideal of a lattice implication algebra and showed that every LI-ideal is a lattice ideal. They gave an example that a lattice ideal may not be an LI-ideal, and showed that every lattice ideal is an LI-ideal in a lattice implication algebra. They discussed the relationship between filters and LI-ideals, and studied how to generate an LI-ideal by a set. Moreover they constructed the quotient structure by using an LI-ideal, and studied the properties of LI-ideals related to implication homomorphisms. In this paper, the notion of a fantastic filter in a lattice implication algebra is introduced, and then we give the relations among filter, positive implicative filter and fantastic filter. We investigate an equivalent condition for a filter to be fantastic, and state an extension property for fantastic filter.

2. Preliminaries. By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution "$'$" and a binary operation "$\to$" satisfying the following axioms:

(I1) $x \to (y \to z) = y \to (x \to z)$,

(I2) $x \to x = 1$,

(I3) $x \to y = y' \to x'$,

(I4) $x \to y = y \to x = 1 \Rightarrow x = y$,

(I5) $(x \to y) \to y = (y \to x) \to x$,

(L1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

(L2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

for all $x, y, z \in L$.

Note that the conditions (L1) and (L2) are equivalent to the conditions

(L3) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, and

(L4) $x \to (y \vee z) = (x \to y) \vee (x \to z)$, respectively.

Example 2.1. Let $L := \{0, a, b, c, 1\}$. Define the partially ordered relation on L as $0 < a < b < c < 1$, and define $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$ for all $x, y \in L$ and "$'$"and "$\to$" as follows:

Table 2.1.

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra.

In the sequel the binary operation "$\to$" will be denoted by juxtaposition. We can define a partial ordering "$\le$" on a lattice implication algebra L by $x \le y$ if and only if $xy = 1$.

In a lattice implication algebra L, the following hold (see [5]):

(1) $0x = 1$, $1x = x$, and $x1 = 1$.

(2) $x' = x0$.

(3) $xy \le (yz)(xz)$.

(4) $x \vee y = (xy)y$.

(5) $((yx)y')' = x \wedge y = ((xy)x')'$.

(6) $x \le y$ implies $yz \le xz$ and $zx \le zy$.

(7) $x \le (xy)y$.

In what follows, L denotes a lattice implication algebra unless otherwise specified.

Definition 2.2 (Xu et al. [7]). A subset F of L is called a *filter* of L if it satisfies:

(F1) $1 \in F$,

(F2) $x \in F$ and $xy \in F$ imply $y \in F$ for all $x, y \in L$.

A subset F of L is called an *implicative filter* of L if it satisfies (F1) and

(F3) $x(yz) \in F$ and $xy \in F$ imply $xz \in F$ for all $x, y, z \in L$.

Proposition 2.3 (Jun [2, Proposition 3.2]). *Every filter F of L has the following property:*

$$x \le y \text{ and } x \in F \text{ imply } y \in F. \tag{2.1}$$

Definition 2.4 (Jun et al. [4]). A subset F of L is called a *positive implicative filter* of L if it satisfies (F1) and

(F4) $x((yz)y) \in F$ and $x \in F$ imply $y \in F$ for all $x, y, z \in L$.

Proposition 2.5 (Jun [4, Theorem 3.1]). *Every positive implicative filter F of L is a filter.*

Proposition 2.6 (Jun [4, Theorem 3.3]). *Let F be a filter of L. Then F is a positive implicative filter of L if and only if*

(F5) $(xy)x \in F$ *implies* $x \in F$ *for all* $x, y \in L$.

Proposition 2.7 (Jun [2, Theorem 3.4]). *Let F be a non-empty subset of L. Then F is a filter of L if and only if it satisfies: for all $x, y \in F$ and $z \in L$,*

(F6) $x \le yz$ *implies* $z \in F$.

3. Fantastic filters

Definition 3.1. A non-empty subset F of L is called a *fantastic filter* of L if it satisfies (F1) and

(F7) $z(yx) \in F$ and $z \in F$ imply $((xy)y)x \in F$ for all $x, y, z \in L$.

Example 3.2. Let $L := \{0, a, b, c, d, 1\}$ be a set with Figure 3.1 as a partial ordering. Define a unary operation "$'$" and a binary operation denoted by juxtaposition on L as follows (Tables 3.2 and 3.3, respectively):

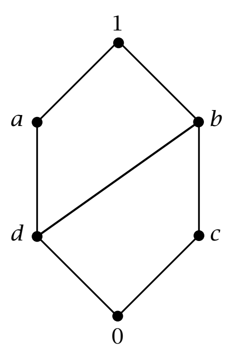

Figure 3.1.

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (xy)y, \qquad x \wedge y := ((x'y')y')', \tag{3.1}$$

TABLE 3.2.

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

TABLE 3.3.

	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

for all $x,y \in L$. Then L is a lattice implication algebra. One can see that $F := \{b,c,1\}$ is a fantastic filter of L.

THEOREM 3.3. *Every fantastic filter of L is a filter.*

PROOF. Let F be a fantastic filter of L and let $zx \in F$ and $z \in F$. Then $z(1x) \in F$ and $z \in F$. It follows from (F7) that $x = ((x1)1)x \in F$ so that F is a filter. □

We now give an equivalent condition for a filter to be a fantastic filter.

THEOREM 3.4. *A filter F of L is fantastic if and only if it satisfies:*
(F8) $yx \in F$ *implies* $((xy)y)x \in F$ *for all* $x,y \in L$.

PROOF. Assume that F is a fantastic filter of L and let $x,y \in L$ be such that $yx \in F$. Then $1(yx) = yx \in F$ and $1 \in F$. It follows from (F7) that $((xy)y)x \in F$. Conversely let F be a filter of L satisfying (F8) and let $x,y,z \in L$ be such that $z(yx) \in F$ and $z \in F$. Then $yx \in F$ by (F2) and hence $((xy)y)x \in F$ by (F8). Therefore F is a fantastic filter of L. □

THEOREM 3.5. *Every positive implicative filter of L is fantastic.*

PROOF. Let F be a positive implicative filter of L. Then F is a filter of L (see Proposition 2.5). Let $x,y \in L$ be such that $yx \in F$. It is sufficient to show that $((xy)y)x \in F$. Since $x \le ((xy)y)x$, we get $(((xy)y)x)y \le xy$. Putting $a = ((xy)y)x$, we obtain

$$\begin{aligned}(ay)a &= ((((xy)y)x)y)(((xy)y)x)\\ &\ge (xy)(((xy)y)x) = ((xy)y)((xy)x) \ge yx.\end{aligned} \tag{3.2}$$

It follows from Proposition 2.3 that $(ay)a \in F$ so, from Proposition 2.6, that $a \in F$, i.e., $((xy)y)x \in F$. Hence F is a fantastic filter of L. □

Open problem. Does the converse of Theorem 3.5 hold?

Theorem 3.6 (extension property for fantastic filter)**.** *Let F and G be filters of L such that $F \subseteq G$. If F is fantastic, then so is G.*

Proof. Let $x, y \in L$ be such that $yx \in G$. Then $y((yx)x) = (yx)(yx) = 1 \in F$. Since F is fantastic, it follows from Theorem 3.4 that

$$((((yx)x)y)y)((yx)x) \in F \tag{3.3}$$

so that $(yx)(((((yx)x)y)y)x) \in F \subseteq G$. Since $yx \in G$, therefore $((((yx)x)y)y)x \in G$. But

$$\begin{aligned}((((((yx)x)y)y)x)(((xy)y)x) &\geq ((xy)y)((((yx)x)y)y) \\ &\geq (((yx)x)y)(xy) \geq x((yx)x) \\ &= (yx)(xx) = (yx)1 = 1.\end{aligned} \tag{3.4}$$

Using Proposition 2.7, we get $((xy)y)x \in G$. Hence, by Theorem 3.4, G is a fantastic filter of L. □

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