ON EXISTENCE OF PERIODIC SOLUTIONS OF THE RAYLEIGH EQUATION OF RETARDED TYPE

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ABSTRACT. In this paper, we give two sufficient conditions on the existence of periodic solutions of the non-autonomous Rayleigh equation of retarded type by using the coincidence degree theory.

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1. Introduction. In [1, 2], the authors studied the existence of periodic solutions of the differential equation
\[ x''(t) + f(x'(t)) + h(t,x(t)) = 0. \] (1.1)
In this paper, we discuss the existence of periodic solutions of the non-autonomous Rayleigh equation of related type
\[ x''(t) + f(t,x'(t-\tau)) + g(t,x(t-\sigma)) = p(t), \] (1.2)
where \( \tau, \sigma \geq 0 \) are constants, \( f \) and \( g \) are functions with period \( 2\pi \) for \( t \), \( f(t,0) = 0 \) for \( t \in R \), \( p \in C(R,R) \), \( p(t) = p(t + 2\pi) \) for \( t \in R \) and \( \int_0^{2\pi} p(t) = 0 \). Using coincidence degree theory developed by Mawhin [2], we find two sufficient conditions for the existence of periodic solutions of (1.2).

2. Main results

THEOREM 2.1. Suppose there are positive constants \( K, D, \) and \( M \) such that
(i) \( |f(t,x)| \leq K \) for \( (t,x) \in R^2 \);
(ii) \( xg(t,x) > 0 \) and \( |g(t,x)| > K \) for \( t \in R \) and \( |x| \geq D \);
(iii) \( g(t,x) \geq -M \) for \( t \in R \) and \( x \leq -D \);
(iv) \( \sup_{(t,x)\in R\times[-D,D]} |g(t,x)| < +\infty \).
Then (1.2) has at least a periodic solution with period \( 2\pi \).

PROOF. Consider the equation
\[ x''(t) + \lambda f(t,x'(t-\tau)) + \lambda g(t,x(t-\sigma)) = \lambda p(t), \] (2.1)
where \( \lambda \in (0,1) \). Suppose that \( x(t) \) is a periodic solution with period \( 2\pi \) of (2.1). Since \( x(0) = x(2\pi) \), there is some \( t_0 \in [0,2\pi] \) such that \( x'(t_0) = 0 \). In view of (2.1), we see
that for any \( t \in [0, 2\pi] \),
\[
|\chi'(t)| = \left| \int_0^t \chi''(s) \, ds \right| \leq \int_0^{2\pi} |\chi''(s)| \, ds
\]
\[
\leq \lambda \int_0^{2\pi} |f(s, \chi'(s - \tau))| \, ds + \lambda \int_0^{2\pi} |g(s, \chi(s - \sigma))| \, ds + \lambda \int_0^{2\pi} |p(s)| \, ds
\]
\[
\leq 2\pi K + \int_0^{2\pi} |g(s, \chi(s - \sigma))| \, ds + 2\pi \max_{0 \leq s \leq 2\pi} |p(s)|.
\]  
(2.2)

We assert that
\[
\int_0^{2\pi} |g(s, \chi(s - \sigma))| \, ds \leq 2\pi K + 4\pi D_1
\]  
(2.3)

for some positive number \( D_1 \). Indeed, integrating (2.1) from 0 to \( 2\pi \) and noting condition (i), we see that
\[
\int_0^{2\pi} \{g(t, \chi(t - \sigma)) - K\} \, dt \leq \int_0^{2\pi} \{g(t, \chi(t - \sigma)) - |f(t, \chi'(t - \tau))|\} \, dt
\]
\[
\leq \int_0^{2\pi} \{f(t, \chi'(t - \tau)) + g(t, \chi(t - \sigma))\} \, dt = 0.
\]  
(2.4)

Thus letting
\[
E_1 = \{t \in [0, 2\pi] \mid \chi(t - \tau) > D\}, \quad E_2 = [0, 2\pi] \setminus E_1.
\]  
(2.5)

By applying (ii), (iii), and (iv), we have
\[
\int_{E_2} |g(t, \chi(t - \sigma))| \, dt \leq 2\pi \max \left\{ M, \sup_{(t, x) \in R \times [-D, D]} |g(t, x)| \right\},
\]  
(2.6)

\[
\int_{E_1} \{ |g(t, \chi(t - \sigma))| - K \} \, dt
\]
\[
\leq \int_{E_1} |g(t, \chi(t - \sigma)) - K| \, dt = \int_{E_1} \{g(t, \chi(t - \sigma)) - K\} \, dt
\]
\[
\leq -\int_{E_2} \{g(t, \chi(t - \sigma)) - K\} \, dt \leq \int_{E_2} |g(t, \chi(t - \sigma))| \, dt + \int_{E_2} K \, dt.
\]  
(2.7)

Therefore
\[
\int_0^{2\pi} |g(t, \chi(t - \sigma))| \, dt \leq 2\pi K + 4\pi \max \left\{ M, \sup_{(t, x) \in R \times [-D, D]} |g(t, x)| \right\},
\]  
(2.8)

and so (2.3) holds. Combining (2.2) and (2.3), we see that
\[
|\chi'(t)| \leq D_2, \quad t \in [0, 2\pi]
\]  
(2.9)

for some positive number \( D_2 \). Next, note that the last equality in (2.4) implies
\[
f(t_1, \chi'(t_1 - \tau)) + g(t_1, \chi(t_1 - \sigma)) = 0
\]  
(2.10)

for some \( t_1 \) in \([0, 2\pi]\). Thus in view of condition (i), we have
\[
|g(t_1, \chi(t_1 - \sigma))| = |f(t_1, \chi'(t_1 - \tau))| \leq K,
\]  
(2.11)

and in view of (ii), we have
\[
|\chi(t_1 - \sigma)| < D.
\]  
(2.12)
Since $x(t)$ is a periodic solution with period $2\pi$ of (2.1), we infer that $|x(t_2)| < D$ for some $t_2$ in $[0, 2\pi]$. Therefore,

$$|x(t)| = |x(t_2) + \int_{t_2}^t x'(t)dt| \leq D + \int_0^{2\pi} |x'(t)| dt \leq D + 2\pi D_2, \quad t \in [0, 2\pi]. \quad (2.13)$$

Let $X$ be the Banach space of all continuous differentiable functions of the form $x = x(t)$, defined on $R$ such that $x(t + 2\pi) = x(t)$ for all $t$, and endowed with the norm $\|x\|_1 = \max_{0 \leq t \leq 2\pi} \{|x(t)|, |x'(t)|\}$. Let $Y$ be the Banach space of all continuous functions of the form $y = y(t)$, defined on $R$ such that $y(t + 2\pi) = y(t)$ for all $t$, and endowed with the norm $\|y\|_0 = \max_{0 \leq t \leq 2\pi} |y(t)|$, and let $\Omega$ be the subspace of $X$ containing functions of the form $x = x(t)$, such that $|x(t)| < \bar{D}$ and $|x'(t)| < \bar{D}$, where $\bar{D}$ is a fixed number greater than $D + 2\pi D_2$. Now, let $L : X \cap C^2(R, R) \to Y$ be the differential operator defined by $(Lx)(t) = x''(t)$ for $t \in R$, and let $N : X \to Y$ be defined by

$$(Nx)(t) = -f(t, x'(t - \sigma)) - g(t, x(t - \tau)) + p(t), \quad t \in R. \quad (2.14)$$

We know that $\ker L = R$. Furthermore if we define the projections $P : X \to \ker L$ and $Q : Y \to Y/\text{Im} L$ by

$$PX = \frac{1}{2\pi} \int_0^{2\pi} x(t)dt,$$

$$Qy = \frac{1}{2\pi} \int_0^{2\pi} y(t)dt,$$

respectively, then $\ker L = \text{Im} P$ and $\ker Q = \text{Im} L$. Furthermore, the operator $L$ is a Fredholm operator with index zero, and the operator $N$ is $L$-compact on the closure $\bar{\Omega}$ of $\Omega$ (see, e.g., [2, p. 176]). In terms of valuation of bound of periodic solutions as above, we know that for any $\lambda \in (0, 1)$ and any $x = x(t)$ in the domain of $L$, which also belongs to $\partial \Omega$, $Lx \neq \lambda Nx$. Since for any $x \in \partial \Omega \cap \ker L$, $x = \bar{D}$ or $x = -\bar{D}$, then in view of (ii), (iii), and $\int_0^{2\pi} p(t)dt = 0$, we have

$$QN x = \frac{1}{2\pi} \int_0^{2\pi} \left[-f(t, x'(t - \tau)) - g(t, x(t - \sigma)) + p(t)\right]dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[-f(t, 0) - g(t, x(t - \sigma))\right]dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[-g(t, x(t - \sigma))\right]dt$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} g(t, x)dt \neq 0. \quad (2.16)$$

In particular, we see that

$$-\frac{1}{2\pi} \int_0^{2\pi} g(t, -\bar{D})dt > 0, \quad (2.17)$$

$$-\frac{1}{2\pi} \int_0^{2\pi} g(t, \bar{D})dt < 0.$$
This shows that
\[
\text{deg} \left[ QNx, \Omega \cap \ker L, 0 \right] \neq 0. \tag{2.18}
\]

In view of Mawhin continuation theorem [2, p. 40], there exists a periodic solution with period \(2\pi\) of (1.2). This completes the proof.

**Theorem 2.2.** Suppose that there are positive constants \(K, D, \text{ and } M\) such that

(i) \(|f(t,x)| \leq K \text{ for } (t,x) \in \mathbb{R}^2\);

(ii) \(xg(t,x) > 0 \text{ and } |g(t,x)| > K \text{ for } t \in \mathbb{R}, |x| \geq D\);

(iii) \(g(t,x) \leq M \text{ for } t \in \mathbb{R}, x \geq D\);

(iv) \(\sup_{(t,x) \in \mathbb{R} \times [-D,D]} |g(t,x)| < +\infty\).

Then (1.2) has at least a periodic solution with period \(2\pi\).

The proof of Theorem 2.2 is similitude of Theorem 2.1, and so, we omit the details here.

**References**


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