Internat. J. Math. & Math. Sci.
Vol. 23, No. 1 (2000) 65–68
S0161171200001836

ON EXISTENCE OF PERIODIC SOLUTIONS OF THE RAYLEIGH EQUATION OF RETARDED TYPE

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(Received 31 July 1998)

Abstract. In this paper, we give two sufficient conditions on the existence of periodic solutions of the non-autonomous Rayleigh equation of retarded type by using the coincidence degree theory.

Keywords and phrases. Rayleigh equation, periodic solution, coincidence degree.

2000 Mathematics Subject Classification. 34K13.

1. Introduction. In [1, 2], the authors studied the existence of periodic solutions of the differential equation

$$x''(t)+f(x'(t))+h(t,x(t))=0. \tag{1.1}$$

In this paper, we discuss the existence of periodic solutions of the non-autonomous Rayleigh equation of related type

$$x''(t)+f(t,x'(t-\tau))+g(t,x(t-\sigma))=p(t), \tag{1.2}$$

where τ, $\sigma \geq 0$ are constants, f and $g \in C(R^2,R)$, $f(t,x)$ and $g(t,x)$ are functions with period 2π for t, $f(t,0)=0$ for $t \in R$, $p \in C(R,R)$, $p(t)=p(t+2\pi)$ for $t \in R$ and $\int_0^{2\pi} p(t)=0$. Using coincidence degree theory developed by Mawhin [2], we find two sufficient conditions for the existence of periodic solutions of (1.2).

2. Main results

Theorem 2.1. *Suppose there are positive constants K, D, and M such that*
(i) $|f(t,x)| \leq K$ *for* $(t,x) \in R^2$;
(ii) $xg(t,x) > 0$ *and* $|g(t,x)| > K$ *for* $t \in R$ *and* $|x| \geq D$;
(iii) $g(t,x) \geq -M$ *for* $t \in R$ *and* $x \leq -D$;
(iv) $\sup_{(t,x)\in R\times[-D,D]} |g(t,x)| < +\infty$.
Then (1.2) has at least a periodic solution with period 2π.

Proof. Consider the equation

$$x''(t)+\lambda f(t,x'(t-\tau))+\lambda g(t,x(t-\sigma))=\lambda p(t), \tag{2.1}$$

where $\lambda \in (0,1)$. Suppose that $x(t)$ is a periodic solution with period 2π of (2.1). Since $x(0)=x(2\pi)$, there is some $t_0 \in [0,2\pi]$ such that $x'(t_0)=0$. In view of (2.1), we see that for any $t \in [0,2\pi]$,

$$\begin{aligned}|x'(t)| &= \left|\int_{t_0}^{t} x''(s)ds\right| \le \int_0^{2\pi} |x''(s)|ds \\ &\le \lambda\int_0^{2\pi} |f(s,x'(s-\tau))|ds + \lambda\int_0^{2\pi} |g(s,x(s-\sigma))|ds + \lambda\int_0^{2\pi} |p(s)|ds \\ &\le 2\pi K + \int_0^{2\pi} |g(s,x(s-\sigma))|ds + 2\pi \max_{0\le s\le 2\pi} |p(s)|.\end{aligned} \tag{2.2}$$

We assert that

$$\int_0^{2\pi} |g(s,x(s-\sigma))|ds \le 2\pi K + 4\pi D_1 \tag{2.3}$$

for some positive number D_1. Indeed, integrating (2.1) from 0 to 2π and noting condition (i), we see that

$$\begin{aligned}\int_0^{2\pi} \left\{g(t,x(t-\sigma)) - K\right\}dt &\le \int_0^{2\pi} \left\{g(t,x(t-\sigma)) - |f(t,x'(t-\tau))|\right\}dt \\ &\le \int_0^{2\pi} \left\{f(t,x'(t-\tau)) + g(t,x(t-\sigma))\right\}dt = 0.\end{aligned} \tag{2.4}$$

Thus letting

$$E_1 = \left\{t \in [0,2\pi] \mid x(t-\sigma) > D\right\}, \qquad E_2 = [0,2\pi]\backslash E_1. \tag{2.5}$$

By applying (ii), (iii), and (iv), we have

$$\int_{E_2} |g(t,x(t-\sigma))|dt \le 2\pi \max\left\{M, \sup_{(t,x)\in R\times[-D,D]} |g(t,x)|\right\}, \tag{2.6}$$

$$\begin{aligned}&\int_{E_1} \left\{|g(t,x(t-\sigma))| - K\right\}dt \\ &\quad\le \int_{E_1} |g(t,x(t-\sigma)) - K|dt = \int_{E_1} \left\{g(t,x(t-\sigma)) - K\right\}dt \\ &\quad\le -\int_{E_2} \left\{g(t,x(t-\sigma)) - K\right\}dt \le \int_{E_2} |g(t,x(t-\sigma))|dt + \int_{E_2} K\,dt.\end{aligned} \tag{2.7}$$

Therefore

$$\int_0^{2\pi} |g(t,x(t-\sigma))|dt \le 2\pi K + 4\pi \max\left\{M, \sup_{(t,x)\in R\times[-D,D]} |g(t,x)|\right\}, \tag{2.8}$$

and so (2.3) holds. Combining (2.2) and (2.3), we see that

$$|x'(t)| \le D_2, \quad t \in [0,2\pi] \tag{2.9}$$

for some positive number D_2. Next, note that the last equality in (2.4) implies

$$f(t_1,x'(t_1-\tau)) + g(t_1,x(t_1-\sigma)) = 0 \tag{2.10}$$

for some t_1 in $[0,2\pi]$. Thus in view of condition (i), we have

$$|g(t_1,x(t_1-\sigma))| = |f(t_1,x'(t_1-\tau))| \le K, \tag{2.11}$$

and in view of (ii), we have

$$|x(t_1-\sigma)| < D. \tag{2.12}$$

Since $x(t)$ is a periodic solution with period 2π of (2.1), we infer that $|x(t_2)| < D$ for some t_2 in $[0,2\pi]$. Therefore,

$$|x(t)| = \left| x(t_2) + \int_{t_2}^{t} x'(t)dt \right| \le D + \int_0^{2\pi} |x'(t)|dt \le D + 2\pi D_2, \quad t \in [0,2\pi]. \tag{2.13}$$

Let X be the Banach space of all continuous differentiable functions of the form $x = x(t)$, defined on R such that $x(t+2\pi) = x(t)$ for all t, and endowed with the norm $\|x\|_1 = \max_{0\le t\le 2\pi}\{|x(t)|, |x'(t)|\}$. Let Y be the Banach space of all continuous functions of the form $y = y(t)$, defined on R such that $y(t+2\pi) = y(t)$ for all t, and endowed with the norm $\|y\|_0 = \max_{0\le t\le 2\pi}|y(t)|$, and let Ω be the subspace of X containing functions of the form $x = x(t)$, such that $|x(t)| < \bar{D}$ and $|x'(t)| < \bar{D}$, where $\bar{D}$ is a fixed number greater than $D + 2\pi D_2$. Now, let $L : X \cap C^{(2)}(R,R) \to Y$ be the differential operator defined by $(Lx)(t) = x''(t)$ for $t \in R$, and let $N : X \to Y$ be defined by

$$(Nx)(t) = -f(t, x'(t-\sigma)) - g(t, x(t-\tau)) + p(t), \quad t \in R. \tag{2.14}$$

We know that $\ker L = R$. Furthermore if we define the projections $P : X \to \ker L$ and $Q : Y \to Y/\operatorname{Im} L$ by

$$\begin{aligned} Px &= \frac{1}{2\pi}\int_0^{2\pi} x(t)dt, \\ Qy &= \frac{1}{2\pi}\int_0^{2\pi} y(t)dt, \end{aligned} \tag{2.15}$$

respectively, then $\ker L = \operatorname{Im} P$ and $\ker Q = \operatorname{Im} L$. Furthermore, the operator L is a Fredholm operator with index zero, and the operator N is L-compact on the closure $\bar{\Omega}$ of Ω (see, e.g., [2, p. 176]). In terms of valuation of bound of periodic solutions as above, we know that for any $\lambda \in (0,1)$ and any $x = x(t)$ in the domain of L, which also belongs to $\partial\Omega$, $Lx \ne \lambda Nx$. Since for any $x \in \partial\Omega \cap \ker L$, $x = \bar{D}$ or $x = -\bar{D}$, then in view of (ii), (iii), and $\int_0^{2\pi} p(t)dt = 0$, we have

$$\begin{aligned} QNx &= \frac{1}{2\pi}\int_0^{2\pi} \left[-f(t, x'(t-\tau)) - g(t, x(t-\sigma)) + p(t) \right] dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left[-f(t,0) - g(t, x(t-\sigma)) \right] dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left[-g(t, x(t-\sigma)) \right] dt \\ &= -\frac{1}{2\pi}\int_0^{2\pi} g(t,x)dt \ne 0. \end{aligned} \tag{2.16}$$

In particular, we see that

$$\begin{aligned} &-\frac{1}{2\pi}\int_0^{2\pi} g(t, -\bar{D})dt > 0, \\ &-\frac{1}{2\pi}\int_0^{2\pi} g(t, \bar{D})dt < 0. \end{aligned} \tag{2.17}$$

This shows that

$$\deg\{QNx, \Omega \cap \ker L, 0\} \neq 0. \tag{2.18}$$

In view of Mawhin continuation theorem [2, p. 40], there exists a periodic solution with period 2π of (1.2). This completes the proof. □

Theorem 2.2. *Suppose that there are positive constants K, D, and M such that*

(i) $|f(t,x)| \le K$ *for* $(t,x) \in R^2$;

(ii) $xg(t,x) > 0$ *and* $|g(t,x)| > K$ *for* $t \in R$, $|x| \ge D$;

(iii) $g(t,x) \le M$ *for* $t \in R$, $x \ge D$;

(iv) $\sup_{(t,x)\in R\times[-D,D]} |g(t,x)| < +\infty$.

Then (1.2) has at least a periodic solution with period 2π.

The proof of Theorem 2.2 is similitude of Theorem 2.1, and so, we omit the details here.

References

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