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MATRIX TRANSFORMATIONS FROM ABSOLUTELY CONVERGENT SERIES TO CONVERGENT SEQUENCES AS GENERAL WEIGHTED MEAN SUMMABILITY METHODS

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ABSTRACT. We prove the necessary and sufficient conditions for an infinity matrix to be a mapping, from absolutely convergent series to convergent sequences, which is treated as general weighted mean summability methods. The results include a classical result by Hardy and another by Moricz and Rhoades as particular cases.

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1. Introduction. A series

$$\sum_{k=0}^{\infty} x_k \tag{1.1}$$

of complex numbers is said to be summable $(C,1)$ if the sequence

$$\frac{1}{n+1}\sum_{k=0}^{n}\sum_{i=0}^{k} x_i, \quad n=0,1,2,\ldots \tag{1.2}$$

converges to a finite limit as $n \to \infty$.

In [1] Hardy proved the following theorem.

THEOREM 1.1. *The series (1.1) is summable $(C,1)$ to a finite number L if and only if the series*

$$\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{x_k}{k+1} \tag{1.3}$$

converges to the same limit.

For a sequence of positive numbers (p_n), let $P_n := \sum_{k=0}^{n} p_n$. A weighted mean matrix $\bar{N}$ is an infinity lower triangular matrix with entries (see [2])

$$a_{nk} := \frac{p_k}{P_n}, \quad k=0,1,2,\ldots,n,\ n=0,1,2,\ldots. \tag{1.4}$$

The series (1.1) is said to be summable $\bar{N}$ if the following sequence:

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k \sum_{i=0}^{k} x_i, \quad n=0,1,2,\ldots, \tag{1.5}$$

converges to a finite limit as $n \to \infty$.

It is clear that summable $(C,1)$ is a special case of summable $\bar{N}$, where

$$p_k = 1, \quad k = 0,1,2,\dots. \tag{1.6}$$

Based on the above idea, Moricz and Rhoades [2] established a result for a broad class of summability methods, which include the method of summability $(C,1)$ as a particular case.

Theorem 1.2. *Let $\bar{N}$ be a weighted mean matrix determined by a sequence (p_n) of positive numbers such that the following conditions are satisfied:*

$$P_n \longrightarrow \infty, \quad \frac{p_n}{P_n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$
$$\sup_{n\geq 0}\left\{\frac{p_{n+1}p_{n-1}}{p_nP_{n+1}} + P_n\sum_{k=n}^{\infty}\frac{1}{P_{n+1}}\left|\frac{p_{k+1}}{p_k} - \frac{p_{k+2}P_k}{p_{k+1}P_{k+2}}\right|\right\} < \infty, \tag{1.7}$$
$$\sup_{n\geq 0}\left\{\frac{p_n}{p_{n+1}} + \frac{1}{P_n}\sum_{k=0}^{n}\left|\frac{p_kP_{k+1}}{p_{k+1}} - \frac{p_{k-1}P_{k-1}}{p_k}\right|\right\} < \infty,$$

with the agreement that

$$p_{-1} = P_{-1} := 0. \tag{1.8}$$

Then the series (1.1) is summable $\bar{N}$ to a finite number L if and only if the series

$$\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{p_n}{P_k}x_k \tag{1.9}$$

converges to the same limit L.

In this paper, we will study the matrix transformations from the space of absolutely convergent series of complex numbers, l_1, to the space of convergent sequences of complex numbers, c. Then we shall establish a more general result for a broader class of weighted mean methods, which includes the method of summable $\bar{N}$ as a particular case if the series (1.1) is absolutely convergent.

2. Matrix transformations from l_1 to c. Let $A = (a_{nk})$ be an infinity matrix with complex entries and let l denote the linear space of complex number sequences. For a sequence $x = (x_n) \in l$, Ax is in l and its entries are given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad n = 0,1,2,\dots \tag{2.1}$$

provided the series converges to a finite complex number.

The following result is well known (see [3, 4]); we list it as a proposition.

Proposition 2.1. *Let $a = (a_k)$ be a sequence of complex numbers. If for every $x = (x_n) \in l_1$, the series*

$$\sum_{k=0}^{\infty} a_k x_k \tag{2.2}$$

converges to a finite complex number, then

$$\sup_{k\geq 0}\{|a_k|\} < \infty. \tag{2.3}$$

From Proposition 2.1, we have the following interesting result.

Proposition 2.2. *Let $a = (a_k)$ be a sequence of complex numbers. If for every $x = (x_n) \in l_1$, the series*

$$\sum_{k=0}^{\infty} a_k x_k \tag{2.4}$$

converges to a finite complex number, then the linear functional f_a defined on l_1 by

$$f_a(x) = \sum_{k=0}^{\infty} a_k x_k \tag{2.5}$$

is a continuous (bounded) linear functional on l_1, such that

$$\|f_a\| = \sup_{k\geq 0}\{|a_k|\}. \tag{2.6}$$

From Proposition 2.1, we know that A is well defined as a mapping from l_1 to l, if and only if

$$\sup_{k\geq 0}\{|a_{nk}|\} < \infty, \quad \text{for } n = 0,1,2,\ldots. \tag{2.7}$$

The following result has been proved in [4] by using functional analysis techniques. It is also proved by summability methods. We list the following theorem without proof.

Theorem 2.3. *Let $A = (a_{nk})$ be an infinity matrix with complex entries. Then A is a mapping from l_1 to c, if and only if the following conditions are satisfied:*

(i) *for every fixed $k = 0,1,2,\ldots$, the sequence (a_{nk}) converges to a finite limit as $n \to \infty$,*

(ii) $\sup_{n,k\geq 0}\{|a_{nk}|\} < \infty$.

Furthermore, if $A = (a_{nk})$ satisfies conditions (i) and (ii), then for every $x = (x_n) \in l_1$, we have

$$\lim_{n\to\infty}(Ax)_n = \lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty}\Big(\lim_{n\to\infty} a_{nk}\Big)x_k. \tag{2.8}$$

The following corollary follows from Theorem 2.3 and (2.8).

Corollary 2.4. *Let $A = (a_{nk})$ be an infinity matrix with complex entries. If A is a mapping from l_1 to c, then the linear operator A is continuous (bounded) linear operator such that*

$$\|A\| = \sup_{n,k\geq 0}\{|a_{nk}|\}. \tag{2.9}$$

3. Applications to summable $(C,1)$ **and summable** $\bar{N}$**.** The following corollary comes immediately from Theorem 2.3, which describes an equivalent reformulation of summability by more general weighted mean methods which are matrix transformations.

Corollary 3.1. *Let* $A=(a_{nk})$, $B=(b_{nk})$ *be two infinity matrices with complex entries. Suppose* A,B *are mapping from* l_1 *to* c, *that is* A,B *satisfying conditions (i), (ii) of Theorem 2.3. Then for every* $x=(x_n)\in l_1$,

$$\lim_{n\to\infty}(Ax)_n=\lim_{n\to\infty}(Bx)_n \tag{3.1}$$

if and only if for every fixed $k=0,1,2,\ldots$,

$$\lim_{n\to\infty}a_{nk}=\lim_{n\to\infty}b_{nk}. \tag{3.2}$$

Proof. Since A,B satisfy conditions (i), (ii) of Theorem 2.3, then from (2.8), we have

$$\lim_{n\to\infty}(Ax)_n=\lim_{n\to\infty}\sum_{k=0}^{\infty}a_{nk}x_k=\sum_{k=0}^{\infty}\Big(\lim_{n\to\infty}a_{nk}\Big)x_k, \tag{3.3}$$

$$\lim_{n\to\infty}(Bx)_n=\lim_{n\to\infty}\sum_{k=0}^{\infty}b_{nk}x_k=\sum_{k=0}^{\infty}\Big(\lim_{n\to\infty}b_{nk}\Big)x_k, \tag{3.4}$$

for any $x=(x_n)\in l_1$. From (2.8) and (3.4), we see that (3.2) implies (3.1). Now, for every fixed $k=0,1,2,\ldots$, define $x=(x_i)$ by

$$x_i=\begin{cases}1, & \text{if } i=k,\\ 0, & \text{if } i\neq k.\end{cases} \tag{3.5}$$

It is clear that $x\in l_1$. Equations (2.8) and (3.4) imply

$$\lim_{n\to\infty}(Ax)_n=\lim_{n\to\infty}a_{nk},\qquad \lim_{n\to\infty}(Bx)_n=\lim_{n\to\infty}b_{nk}. \tag{3.6}$$

From (3.6), we see that (3.1) implies (3.2). □

Recall that for a sequence of positive numbers (p_n), $P_n=\sum_{k=0}^{n}p_k$. The series (1.1) is said to be summable $\bar{N}$ if the following sequence:

$$\frac{1}{P_n}\sum_{k=n}^{n}p_k\sum_{i=0}^{k}x_i,\quad n=0,1,2,\ldots \tag{3.7}$$

converges to a finite limit as $n\to\infty$.

To generalize Theorem 1.2, we shall construct two weighted mean matrices according to the summability (3.7) and the following summability method:

$$\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{p_n}{P_k}x_k. \tag{3.8}$$

Based on the sequence of positive numbers (p_n), define two infinity matrices $A = (a_{nk})$ and $B = (b_{nk})$, by

$$a_{nk} = \begin{cases} 0, & \text{if } k > n, \\ \dfrac{P_n - P_{k-1}}{P_n}, & \text{if } k \le n, \end{cases} \tag{3.9}$$

$$b_{nk} = \begin{cases} \dfrac{P_n}{P_k}, & \text{if } k > n, \\ 1, & \text{if } k \le n, \end{cases} \tag{3.10}$$

where we agree that $P_{-1} = 0$.

It can be seen that any sequence of positive numbers (p_n), $B = (b_{nk})$ defined by (3.10), always satisfies the conditions (i) and (ii) of Theorem 2.3 and $A = (a_{nk})$ defined by (3.9) always satisfies the conditions (ii) of Theorem 2.3. Furthermore, $A = (a_{nk})$ will satisfies the conditions (i) of Theorem 2.3 if the sequence (p_n) satisfies

$$P_n \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{3.11}$$

Hence we have the following corollary of Theorem 2.3.

Corollary 3.2. *For any sequence of positive numbers (p_n), $B = (b_{nk})$ defined by (3.10) is always a mapping from l_1 to c. If (p_n) satisfying (3.11), then $A = (a_{nk})$ defined by (3.9) is a mapping from l_1 to c.*

The following corollary will give the Moricz and Rhoades's result, Theorem 1.2, if the series (1.1) is absolutely convergent.

Corollary 3.3. *Let (p_n) be a sequence of positive numbers satisfying (3.11). Let $A = (a_{nk})$, $B = (b_{nk})$ be defined by (3.9) and (3.10). Then for any $x = (x_n) \in l_1$, we have*

$$\lim_{n\to\infty} (Ax)_n = \lim_{n\to\infty} (Bx)_n = \sum_{k=0}^{\infty} x_k. \tag{3.12}$$

Proof. Notice that under condition (3.11), we have that for every fixed $k = 0, 1, 2, \ldots$,

$$\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} b_{nk} = 1. \tag{3.13}$$

Then the proof of this corollary follows Corollary 3.2 and the equalities (2.8) and (3.4) immediately. □

From the definitions (3.9) and (3.10), we see that for every fixed $n = 0, 1, 2, \ldots$,

$$(Ax)_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k \sum_{i=0}^{k} x_i, \qquad (Bx)_n = \sum_{m=0}^{n} \sum_{k=m}^{\infty} \frac{p_n}{P_k} x_k. \tag{3.14}$$

Corollary 3.3 shows that if the sequence of positive numbers (p_n) satisfies condition (3.11), then for any $x = (x_n) \in l_1$, we have

$$\lim_{n\to\infty} \frac{1}{P_n} \sum_{k=0}^{n} p_k \sum_{i=0}^{k} x_i = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{p_n}{P_k} x_k = \sum_{k=0}^{\infty} x_k. \tag{3.15}$$

In a particular case, as mentioned by Moricz and Rhoades [2], taking $p_k = 1$, for $k = 0, 1, 2, \ldots$, we find the Hardy's result, Theorem 1.1, if that the series (1.1) is absolutely convergent, that is, for any $x = (x_n) \in l_1$,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \sum_{i=0}^{k} x_i = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{x_k}{k+1} = \sum_{k=0}^{\infty} x_k. \tag{3.16}$$

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