

## MATRIX TRANSFORMATIONS FROM ABSOLUTELY CONVERGENT SERIES TO CONVERGENT SEQUENCES AS GENERAL WEIGHTED MEAN SUMMABILITY METHODS

JINLU LI

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**ABSTRACT.** We prove the necessary and sufficient conditions for an infinity matrix to be a mapping, from absolutely convergent series to convergent sequences, which is treated as general weighted mean summability methods. The results include a classical result by Hardy and another by Moricz and Rhoades as particular cases.

**Keywords and phrases.** Weighted mean matrix, matrix transformation.

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**1. Introduction.** A series

$$\sum_{k=0}^{\infty} x_k \quad (1.1)$$

of complex numbers is said to be summable  $(C, 1)$  if the sequence

$$\frac{1}{n+1} \sum_{k=0}^n \sum_{i=0}^k x_i, \quad n = 0, 1, 2, \dots \quad (1.2)$$

converges to a finite limit as  $n \rightarrow \infty$ .

In [1] Hardy proved the following theorem.

**THEOREM 1.1.** *The series (1.1) is summable  $(C, 1)$  to a finite number  $L$  if and only if the series*

$$\sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{x_k}{k+1} \quad (1.3)$$

*converges to the same limit.*

For a sequence of positive numbers  $(p_n)$ , let  $P_n := \sum_{k=0}^n p_k$ . A weighted mean matrix  $\bar{N}$  is an infinity lower triangular matrix with entries (see [2])

$$a_{nk} := \frac{p_k}{P_n}, \quad k = 0, 1, 2, \dots, n, \quad n = 0, 1, 2, \dots \quad (1.4)$$

The series (1.1) is said to be summable  $\bar{N}$  if the following sequence:

$$\frac{1}{P_n} \sum_{k=0}^n p_k \sum_{i=0}^k x_i, \quad n = 0, 1, 2, \dots, \quad (1.5)$$

converges to a finite limit as  $n \rightarrow \infty$ .

It is clear that summable  $(C, 1)$  is a special case of summable  $\tilde{N}$ , where

$$p_k = 1, \quad k = 0, 1, 2, \dots \tag{1.6}$$

Based on the above idea, Moricz and Rhoades [2] established a result for a broad class of summability methods, which include the method of summability  $(C, 1)$  as a particular case.

**THEOREM 1.2.** *Let  $\tilde{N}$  be a weighted mean matrix determined by a sequence  $(p_n)$  of positive numbers such that the following conditions are satisfied:*

$$\begin{aligned} &P_n \rightarrow \infty, \quad \frac{p_n}{P_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ &\sup_{n \geq 0} \left\{ \frac{p_{n+1}p_{n-1}}{p_n P_{n+1}} + P_n \sum_{k=n}^{\infty} \frac{1}{P_{n+1}} \left| \frac{p_{k+1}}{p_k} - \frac{p_{k+2}P_k}{p_{k+1}P_{k+2}} \right| \right\} < \infty, \\ &\sup_{n \geq 0} \left\{ \frac{p_n}{p_{n+1}} + \frac{1}{P_n} \sum_{k=0}^n \left| \frac{p_k P_{k+1}}{p_{k+1}} - \frac{p_{k-1}P_{k-1}}{p_k} \right| \right\} < \infty, \end{aligned} \tag{1.7}$$

with the agreement that

$$p_{-1} = P_{-1} := 0. \tag{1.8}$$

Then the series (1.1) is summable  $\tilde{N}$  to a finite number  $L$  if and only if the series

$$\sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{p_n}{P_k} x_k \tag{1.9}$$

converges to the same limit  $L$ .

In this paper, we will study the matrix transformations from the space of absolutely convergent series of complex numbers,  $l_1$ , to the space of convergent sequences of complex numbers,  $c$ . Then we shall establish a more general result for a broader class of weighted mean methods, which includes the method of summable  $\tilde{N}$  as a particular case if the series (1.1) is absolutely convergent.

**2. Matrix transformations from  $l_1$  to  $c$ .** Let  $A = (a_{nk})$  be an infinity matrix with complex entries and let  $l$  denote the linear space of complex number sequences. For a sequence  $x = (x_n) \in l$ ,  $Ax$  is in  $l$  and its entries are given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad n = 0, 1, 2, \dots \tag{2.1}$$

provided the series converges to a finite complex number.

The following result is well known (see [3, 4]); we list it as a proposition.

**PROPOSITION 2.1.** *Let  $a = (a_k)$  be a sequence of complex numbers. If for every  $x = (x_n) \in l_1$ , the series*

$$\sum_{k=0}^{\infty} a_k x_k \tag{2.2}$$

converges to a finite complex number, then

$$\sup_{k \geq 0} \{ |a_k| \} < \infty. \tag{2.3}$$

From Proposition 2.1, we have the following interesting result.

**PROPOSITION 2.2.** *Let  $a = (a_k)$  be a sequence of complex numbers. If for every  $x = (x_n) \in l_1$ , the series*

$$\sum_{k=0}^{\infty} a_k x_k \tag{2.4}$$

converges to a finite complex number, then the linear functional  $f_a$  defined on  $l_1$  by

$$f_a(x) = \sum_{k=0}^{\infty} a_k x_k \tag{2.5}$$

is a continuous (bounded) linear functional on  $l_1$ , such that

$$\|f_a\| = \sup_{k \geq 0} \{ |a_k| \}. \tag{2.6}$$

From Proposition 2.1, we know that  $A$  is well defined as a mapping from  $l_1$  to  $l$ , if and only if

$$\sup_{k \geq 0} \{ |a_{nk}| \} < \infty, \quad \text{for } n = 0, 1, 2, \dots \tag{2.7}$$

The following result has been proved in [4] by using functional analysis techniques. It is also proved by summability methods. We list the following theorem without proof.

**THEOREM 2.3.** *Let  $A = (a_{nk})$  be an infinity matrix with complex entries. Then  $A$  is a mapping from  $l_1$  to  $c$ , if and only if the following conditions are satisfied:*

- (i) *for every fixed  $k = 0, 1, 2, \dots$ , the sequence  $(a_{nk})$  converges to a finite limit as  $n \rightarrow \infty$ ,*
- (ii)  $\sup_{n, k \geq 0} \{ |a_{nk}| \} < \infty$ .

Furthermore, if  $A = (a_{nk})$  satisfies conditions (i) and (ii), then for every  $x = (x_n) \in l_1$ , we have

$$\lim_{n \rightarrow \infty} (Ax)_n = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left( \lim_{n \rightarrow \infty} a_{nk} \right) x_k. \tag{2.8}$$

The following corollary follows from Theorem 2.3 and (2.8).

**COROLLARY 2.4.** *Let  $A = (a_{nk})$  be an infinity matrix with complex entries. If  $A$  is a mapping from  $l_1$  to  $c$ , then the linear operator  $A$  is continuous (bounded) linear operator such that*

$$\|A\| = \sup_{n, k \geq 0} \{ |a_{nk}| \}. \tag{2.9}$$

**3. Applications to summable  $(C, 1)$  and summable  $\bar{N}$ .** The following corollary comes immediately from Theorem 2.3, which describes an equivalent reformulation of summability by more general weighted mean methods which are matrix transformations.

**COROLLARY 3.1.** *Let  $A = (a_{nk}), B = (b_{nk})$  be two infinity matrices with complex entries. Suppose  $A, B$  are mapping from  $l_1$  to  $c$ , that is  $A, B$  satisfying conditions (i), (ii) of Theorem 2.3. Then for every  $x = (x_n) \in l_1$ ,*

$$\lim_{n \rightarrow \infty} (Ax)_n = \lim_{n \rightarrow \infty} (Bx)_n \tag{3.1}$$

if and only if for every fixed  $k = 0, 1, 2, \dots$ ,

$$\lim_{n \rightarrow \infty} a_{nk} = \lim_{n \rightarrow \infty} b_{nk}. \tag{3.2}$$

**PROOF.** Since  $A, B$  satisfy conditions (i), (ii) of Theorem 2.3, then from (2.8), we have

$$\lim_{n \rightarrow \infty} (Ax)_n = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left( \lim_{n \rightarrow \infty} a_{nk} \right) x_k, \tag{3.3}$$

$$\lim_{n \rightarrow \infty} (Bx)_n = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} b_{nk} x_k = \sum_{k=0}^{\infty} \left( \lim_{n \rightarrow \infty} b_{nk} \right) x_k, \tag{3.4}$$

for any  $x = (x_n) \in l_1$ . From (2.8) and (3.4), we see that (3.2) implies (3.1). Now, for every fixed  $k = 0, 1, 2, \dots$ , define  $x = (x_i)$  by

$$x_i = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases} \tag{3.5}$$

It is clear that  $x \in l_1$ . Equations (2.8) and (3.4) imply

$$\lim_{n \rightarrow \infty} (Ax)_n = \lim_{n \rightarrow \infty} a_{nk}, \quad \lim_{n \rightarrow \infty} (Bx)_n = \lim_{n \rightarrow \infty} b_{nk}. \tag{3.6}$$

From (3.6), we see that (3.1) implies (3.2). □

Recall that for a sequence of positive numbers  $(p_n), P_n = \sum_{k=0}^n p_k$ . The series (1.1) is said to be summable  $\bar{N}$  if the following sequence:

$$\frac{1}{P_n} \sum_{k=n}^n p_k \sum_{i=0}^k x_i, \quad n = 0, 1, 2, \dots \tag{3.7}$$

converges to a finite limit as  $n \rightarrow \infty$ .

To generalize Theorem 1.2, we shall construct two weighted mean matrices according to the summability (3.7) and the following summability method:

$$\sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{p_n}{P_k} x_k. \tag{3.8}$$

Based on the sequence of positive numbers  $(p_n)$ , define two infinity matrices  $A = (a_{nk})$  and  $B = (b_{nk})$ , by

$$a_{nk} = \begin{cases} 0, & \text{if } k > n, \\ \frac{P_n - P_{k-1}}{P_n}, & \text{if } k \leq n, \end{cases} \tag{3.9}$$

$$b_{nk} = \begin{cases} \frac{P_n}{P_k}, & \text{if } k > n, \\ 1, & \text{if } k \leq n, \end{cases} \tag{3.10}$$

where we agree that  $P_{-1} = 0$ .

It can be seen that any sequence of positive numbers  $(p_n)$ ,  $B = (b_{nk})$  defined by (3.10), always satisfies the conditions (i) and (ii) of Theorem 2.3 and  $A = (a_{nk})$  defined by (3.9) always satisfies the conditions (ii) of Theorem 2.3. Furthermore,  $A = (a_{nk})$  will satisfy the conditions (i) of Theorem 2.3 if the sequence  $(p_n)$  satisfies

$$P_n \rightarrow \infty \quad \text{as } n \rightarrow \infty. \tag{3.11}$$

Hence we have the following corollary of Theorem 2.3.

**COROLLARY 3.2.** *For any sequence of positive numbers  $(p_n)$ ,  $B = (b_{nk})$  defined by (3.10) is always a mapping from  $l_1$  to  $c$ . If  $(p_n)$  satisfying (3.11), then  $A = (a_{nk})$  defined by (3.9) is a mapping from  $l_1$  to  $c$ .*

The following corollary will give the Moricz and Rhoades's result, Theorem 1.2, if the series (1.1) is absolutely convergent.

**COROLLARY 3.3.** *Let  $(p_n)$  be a sequence of positive numbers satisfying (3.11). Let  $A = (a_{nk})$ ,  $B = (b_{nk})$  be defined by (3.9) and (3.10). Then for any  $x = (x_n) \in l_1$ , we have*

$$\lim_{n \rightarrow \infty} (Ax)_n = \lim_{n \rightarrow \infty} (Bx)_n = \sum_{k=0}^{\infty} x_k. \tag{3.12}$$

**PROOF.** Notice that under condition (3.11), we have that for every fixed  $k = 0, 1, 2, \dots$ ,

$$\lim_{n \rightarrow \infty} a_{nk} = \lim_{n \rightarrow \infty} b_{nk} = 1. \tag{3.13}$$

Then the proof of this corollary follows Corollary 3.2 and the equalities (2.8) and (3.4) immediately. □

From the definitions (3.9) and (3.10), we see that for every fixed  $n = 0, 1, 2, \dots$ ,

$$(Ax)_n = \frac{1}{P_n} \sum_{k=0}^n p_k \sum_{i=0}^k x_i, \quad (Bx)_n = \sum_{m=0}^n \sum_{k=m}^{\infty} \frac{p_n}{P_k} x_k. \tag{3.14}$$

Corollary 3.3 shows that if the sequence of positive numbers  $(p_n)$  satisfies condition (3.11), then for any  $x = (x_n) \in l_1$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{P_n} \sum_{k=0}^n p_k \sum_{i=0}^k x_i = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{p_n}{P_k} x_k = \sum_{k=0}^{\infty} x_k. \tag{3.15}$$

In a particular case, as mentioned by Moricz and Rhoades [2], taking  $p_k = 1$ , for  $k = 0, 1, 2, \dots$ , we find the Hardy's result, Theorem 1.1, if that the series (1.1) is absolutely convergent, that is, for any  $x = (x_n) \in l_1$ ,

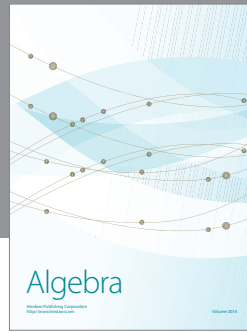
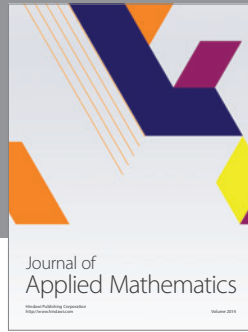
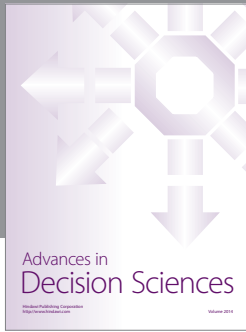
$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n \sum_{i=0}^k x_i = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{x_k}{k+1} = \sum_{k=0}^{\infty} x_k. \quad (3.16)$$

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JINLU LI: DEPARTMENT OF MATHEMATICS, SHAWNEE STATE UNIVERSITY, PORTSMOUTH, OHIO 45662, USA



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