

ON CENTRAL COMMUTATOR GALOIS EXTENSIONS OF RINGS

GEORGE SZETO and LIANYONG XUE

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ABSTRACT. Let B be a ring with 1, G a finite automorphism group of B of order n for some integer n , B^G the set of elements in B fixed under each element in G , and $\Delta = V_B(B^G)$ the commutator subring of B^G in B . Then the type of central commutator Galois extensions is studied. This type includes the types of Azumaya Galois extensions and Galois H -separable extensions. Several characterizations of a central commutator Galois extension are given. Moreover, it is shown that when G is inner, B is a central commutator Galois extension of B^G if and only if B is an H -separable projective group ring $B^G G_f$. This generalizes the structure theorem for central Galois algebras with an inner Galois group proved by DeMeyer.

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1. Introduction. Galois theory for commutative rings were studied in the sixties and seventies (see [4, Chapter 3]), and several Galois extensions of noncommutative rings were also investigated (see [2, 5, 6, 8]). Recently, central Galois extensions and the DeMeyer-Kanzaki Galois extensions were generalized to the Azumaya Galois extensions and center Galois extensions, respectively (see [1, 9, 10, 11]). B is called an Azumaya Galois extension of B^G with Galois group G if B is a Galois extension of B^G which is an Azumaya algebra over C^G where C is the center of B , and B is called a center Galois extension of B^G if C is a Galois algebra with Galois group $G|_C \cong G$. The purpose of the present paper is to study a type of Galois extensions which is strictly between the types of Azumaya Galois extensions and Galois H -separable extensions. Let $\Delta = V_B(B^G)$, the commutator subring of B^G in B . We call B a commutator Galois extension of B^G if Δ is a Galois extension with Galois group $G|_\Delta \cong G$, and B is a central commutator Galois extension of B^G if Δ is a central Galois algebra with Galois group $G|_\Delta \cong G$. We shall characterize a central commutator Galois extension in terms of a Galois H -separable extension B of B^G as studied by Sugano (see [8]) and the C -modules $\{J_g \mid g \in G\}$ where $J_g = \{b \in B \mid ba = g(a)b \text{ for all } a \in B\}$. Moreover, it will be shown that B is a central commutator Galois extension of B^G with an inner Galois group G if and only if B is an H -separable projective group ring $B^G G_f$ where $B^G G_f = \sum_{g \in G} B^G U_g$ such that $\{U_g \mid g \in G\}$ are free over B^G , $bU_g = U_g b$ for all $b \in B^G$ and $g \in G$, and $U_g U_h = U_{gh} f(g, h)$ where $f : G \times G \rightarrow \text{units of } C^G$ is a factor set. This generalizes the structure theorem for a central Galois algebra with an inner Galois group proved by DeMeyer (see [3]).

2. Basic definitions and notation. Throughout this paper, B will represent a ring with 1, C the center of B , G a finite automorphism group of B of order n for some integer n , B^G the set of elements in B fixed under each element in G , and $\Delta = V_B(B^G)$, the commutator subring of B^G in B .

Let A be a subring of a ring B with the same identity 1. We call B a separable extension of A if there exist $\{a_i, b_i$ in B , $i = 1, 2, \dots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where \otimes is over A , and a ring B is called an H -separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B -bimodule. An Azumaya algebra is a separable extension of its center. B is called a Galois extension of B^G with Galois group G if there exist elements $\{c_i, d_i$ in B , $i = 1, 2, \dots, m\}$ for some integer m such that $\sum_{i=1}^m c_i g(d_i) = \delta_{1,g}$ for $g \in G$. The set $\{c_i, d_i\}$ is called a G -Galois system for B . B is called a DeMeyer-Kanzaki Galois extension of B^G if B is an Azumaya C -algebra and C is a Galois algebra with Galois group $G|_C \cong G$. If C is a Galois algebra with Galois group $G|_C \cong G$, we call B a center Galois extension of B^G . B is called an Azumaya Galois extension if it is a Galois extension of B^G that is an Azumaya C^G -algebra, and B is called a Galois H -separable extension if it is a Galois and an H -separable extension of B^G (see [8]). We call B a commutator Galois extension of B^G if Δ is a Galois extension with Galois group $G|_\Delta \cong G$, and B is a central commutator Galois extension of B^G if Δ is a central Galois algebra with Galois group $G|_\Delta \cong G$. For each $g \in G$, let $J_g = \{b \in B \mid bx = g(x)b \text{ for all } x \in B\}$ and $J_g^A = \{a \in A \mid ax = g(x)a \text{ for all } x \in A\}$ for a subring A of B .

3. Central commutator Galois extensions. In this section, we shall give several characterizations of a central commutator Galois extension in terms of Galois H -separable extensions and Azumaya Galois extensions, respectively, and prove the converse of a theorem for a Galois H -separable extension as given in [8]. We begin with some properties of a commutator Galois extension.

LEMMA 3.1. *If B is a commutator Galois extension of B^G , then Δ is a Galois algebra over C^G .*

PROOF. Since Δ is a Galois extension of Δ^G with Galois group $G|_\Delta \cong G$, B and $B^G \Delta$ are also Galois extensions of B^G with Galois group G and $G|_{B^G \Delta}$. But $B^G \Delta \subset B$ and $G \cong G|_{B^G \Delta}$, so $B = B^G \Delta$. Thus, the center of Δ is C ; and so $\Delta^G = B^G \cap \Delta = C^G$. \square

LEMMA 3.2. *If B is a commutator Galois extension of B^G , then $J_g = J_g^\Delta$ for each $g \in G$.*

PROOF. Since $J_g = \{b \in B \mid ba = g(a)b \text{ for all } a \in B\} \subset \{b \in B \mid ba = g(a)b \text{ for all } a \in B^G\} = \Delta$, $J_g \subset J_g^\Delta$.

Conversely, for any $x \in J_g^\Delta$, $xd = g(d)x$ for all $d \in \Delta$. Since Δ is a Galois extension of Δ^G with Galois group $G|_\Delta \cong G$, $B = B^G \Delta$ by the proof of Lemma 3.1. So for any $b \in B$, $b = \sum_{i=1}^m b_i d_i$ for some $b_i \in B^G$, $d_i \in \Delta$ and some integer m , we have that $xb = x \sum_{i=1}^m b_i d_i = \sum_{i=1}^m b_i x d_i = \sum_{i=1}^m b_i g(d_i) x = g(\sum_{i=1}^m b_i d_i) x = g(b)x$. Thus, $J_g^\Delta \subset J_g$; and so $J_g = J_g^\Delta$. \square

THEOREM 3.3. *The following are equivalent:*

- (1) B is a central commutator Galois extension of B^G .

- (2) B is a commutator Galois extension of B^G and $J_g J_{g^{-1}} = C$ for each $g \in G$.
- (3) B is a Galois H -separable extension of B^G , $B = B^G \Delta$, and $n^{-1} \in B$.

PROOF. (1) \Rightarrow (2). It is clear.

(2) \Rightarrow (1). By Lemma 3.1, $\Delta^G = C^G$, so Δ is a Galois algebra with Galois group $G|_{\Delta} \cong G$. By hypothesis, $J_g J_{g^{-1}} = C$ for each $g \in G$ and by Lemma 3.2, $J_g = J_g^{\Delta}$ for each $g \in G$, so Δ is a central Galois algebra (see [5, Theorem 1]).

(1) \Rightarrow (3). Since Δ is a central Galois C^G -algebra, we have $B = B^G \Delta$, $J_g = J_g^{\Delta}$ for each $g \in G$ by Lemma 3.2 and $J_g^{\Delta} J_{g^{-1}}^{\Delta} = C$ (see [6, Lemma 2]). Hence $J_g J_{g^{-1}} = C$ for each $g \in G$. But B is a Galois extension of B^G with the same Galois system for Δ , so B is a Galois H -separable extension of B^G (see [8, Theorem 2(iii) \Rightarrow (i)]). Moreover, $n^{-1} \in B$ (see [6, Corollary 3]), so (3) holds.

(3) \Rightarrow (1). Since $B = B^G \Delta$, the group $H = \{g \in G | g|_{\Delta} \text{ is an identity}\} = \{1\}$. Thus, Δ is a central Galois algebra over Δ^G (see [8, Theorem 6, (3)(ii) \Rightarrow (iii)]) where $\Delta^G = C^G$ by Lemma 3.1.

We remark that (1) \Rightarrow (3) in Theorem 3.3 is the converse of [8, Theorem 6]; that is, if Δ is a central Galois algebra with Galois group $G|_{\Delta} \cong G$, then

- (i) $n^{-1} \in B$,
- (ii) $B = B^G \Delta$,
- (iii) B is a Galois H -separable extension of B^G . □

In the next theorem, we give a characterization of a central commutator Galois extension in terms of Azumaya Galois extensions.

THEOREM 3.4. *The following are equivalent:*

- (1) B is a central commutator Galois extension of B^G and B^G is a separable C^G -algebra.
- (2) B is an Azumaya Galois extension with Galois group G .
- (3) B is a central commutator Galois extension and a separable extension of Δ .

PROOF. (1) \Rightarrow (2). Since B is a central commutator Galois extension, B is a Galois H -separable extension of B^G by Theorem 3.3(3). Thus, $V_B(V_B(B^G)) = B^G$ (see [8, Proposition 4(1)]). This implies that $C \subset B^G$; and so $C = C^G$. Moreover, by Theorem 3.3(3) again, $B = B^G \Delta$, so the center of B^G is C^G , the center of B . Thus, B^G is an Azumaya C^G -algebra. By noting that B is a Galois extension of B^G , (2) holds.

(2) \Rightarrow (1). It is a consequence of [1, Lemma 1].

(1) \Rightarrow (3). Since B is a separable extension of B^G (for it is a Galois extension) and B^G is a separable C^G -algebra, B is a separable C^G -algebra by the transitivity property of separable extensions. Thus, B is a separable extension of Δ because $C^G \subset \Delta \subset B$.

(3) \Rightarrow (1). Since Δ is a Galois extension of Δ^G with Galois group $G|_{\Delta} \cong G$, Δ is a separable extension of Δ^G . By Lemma 3.1, $\Delta^G = C^G = C$ (for C is the center of Δ). By hypothesis, B is a separable extension of Δ . Hence B is a separable extension of C , that is, B is an Azumaya C -algebra. By Lemma 3.1 again, $B = B^G \Delta$ such that B^G and Δ are C -subalgebras of the Azumaya C -algebra B . Hence, they are Azumaya C -algebras by the commutator theorem for Azumaya algebras (see [4, Theorem 4.3, page 57]). Since Δ is a Galois extension of Δ^G with Galois group $G|_{\Delta} \cong G$, B is a Galois extension of B^G which is an Azumaya C^G -algebra. This completes the proof. □

4. H -separable projective group rings. In [3], it was shown that B is a central Galois algebra with an inner Galois group G if and only if B is an Azumaya projective group algebra $C^G G_f$ over C^G where $C^G G_f = \sum_{g \in G} C^G U_g$ such that $\{U_g \mid g \in G\}$ are free over C^G , $cU_g = U_g c$ for all $c \in C^G$ and $g \in G$, and $U_g U_h = U_{gh} f(g, h)$, $f : G \times G \rightarrow$ units of C^G is a factor set (see [3]). We shall generalize this fact to a central commutator Galois extension with an inner Galois group.

THEOREM 4.1. *B is a central commutator Galois extension of B^G with an inner Galois group G if and only if $B = B^G G_f$ which is an H -separable extension of B^G and $n^{-1} \in B$.*

PROOF. (\Rightarrow) By Theorem 3.3 (1) \Rightarrow (3), $B = B^G \Delta$ which is a Galois H -separable extension of B^G and $n^{-1} \in B$, so it suffices to show that $B = B^G G_f$, a projective group ring with coefficient ring B^G . Since Δ is a central Galois C^G -algebra, by [3, Theorem 2], $\Delta = C^G G_f$, a projective group algebra over C^G where $f : G \times G \rightarrow$ units of C^G is a factor set such that $f(g, h) = U_g U_h U_{gh}^{-1}$ for all $g, h \in G$. Noting that $bU_g = U_g b$ for all $b \in B^G$ and $g \in G$, we claim that $\{U_g \mid g \in G\}$ are independent over B^G . Assume $\sum_{g \in G} b_g U_g = 0$ for some $b_g \in B^G$ and $g \in G$. Since Δ is a Galois extension of Δ^G with Galois group $G \mid_{\Delta} \cong G$, there exists a G -Galois system $\{c_i, d_i, i = 1, 2, \dots, m$ for some integer $m\}$ for Δ such that $\sum_{i=1}^m c_i g(d_i) = \delta_{1,g}$ for $g \in G$. Hence

$$\begin{aligned} b_1 &= \sum_{g \in G} \delta_{1,g} b_g U_g = \sum_{g \in G} \sum_{i=1}^m c_i g(d_i) b_g U_g \\ &= \sum_{g \in G} \sum_{i=1}^m c_i b_g g(d_i) U_g = \sum_{g \in G} \sum_{i=1}^m c_i b_g U_g d_i \\ &= \sum_{i=1}^m c_i \left(\sum_{g \in G} b_g U_g \right) d_i = 0. \end{aligned} \tag{4.1}$$

So $\sum_{g \in G} b_g U_g = 0$ for some $b_g \in B^G$ and $g \in G$ implies that $b_1 = 0$. Now for any $h \in G$, since $\sum_{g \in G} b_g U_g = 0$, $0 = \sum_{g \in G} b_g U_g U_{h^{-1}} = \sum_{g \in G} b_g f(g, h^{-1}) U_{gh^{-1}}$. Thus, $b_h f(h, h^{-1}) = 0$, and so $b_h = 0$. This proves that $\{U_g \mid g \in G\}$ are independent over B^G .

(\Leftarrow) Since $B^G G_f (\cong B^G \otimes_{C^G} C^G G_f)$ is an H -separable extension of B^G and B^G is a direct summand of $B^G G_f$ as a left B^G -module, $V_{B^G G_f}(V_{B^G G_f}(B^G)) = B^G$. This implies that the center of $B^G G_f$ is C^G . Moreover, G is inner induced by $\{U_g \mid g \in G\}$, so $J_g = C^G U_g$ for each $g \in G$. But then $C^G G_f = \oplus \sum_{g \in G} C^G U_g = \oplus \sum_{g \in G} J_g$ such that $J_g J_{g^{-1}} = (C^G U_g)(C^G U_{g^{-1}}) = C^G$ for all $g \in G$. By hypothesis, $n^{-1} \in C^G$, so $C^G G_f$ is a separable algebra over C^G . Thus, $\Delta (= C^G G_f)$ is a central Galois algebra (see [5, Theorem 1]) with an inner Galois group \bar{G} induced by $\{U_g \mid g \in G\}$. Thus, B is a central commutator Galois extension of B^G with an inner Galois group G .

By [7, Theorem 1.2], we derive a one-to-one correspondence between some sets of separable subextensions in a central commutator Galois extension B of B^G . Let $\mathcal{S} = \{\mathcal{A} \mid \mathcal{A} \text{ is a separable subextension of } B \text{ containing } B^G \text{ which is a direct summand of } B \text{ as a bimodule}\}$ and $\mathcal{T} = \{\mathcal{D} \mid \mathcal{D} \text{ is a separable subalgebra of } \Delta \text{ over } C^G\}$. \square

THEOREM 4.2. *Let B be a central commutator Galois extension of B^G . Then, there exists a one-to-one correspondence between \mathcal{S} and \mathcal{T} by $A \rightarrow V_B(A)$.*

PROOF. By Theorem 3.3(3), B is an H -separable extension of B^G , so the correspondence holds by [7, Theorem 1.2]. □

We conclude this paper with two examples of Galois extension B to show that

- (1) B is a central commutator Galois extension but not an Azumaya Galois extension (see Theorem 3.4),
- (2) B is a Galois H -separable extension but not a central commutator Galois extension (see Theorem 3.3).

EXAMPLE 4.3. Let $A = Q[i, j, k]$ be the quaternion algebra over the rational field Q , $B = \left\{ \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \mid a_1, a_2, a_3 \in A \right\}$, the ring of all 2-by-2 upper triangular matrices over A and $G = \{1, g_i, g_j, g_k\}$ where $g_i(a) = iai^{-1}$, $g_j(a) = jaj^{-1}$, $g_k(a) = kak^{-1}$ for all a in A and $g \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} = \begin{pmatrix} g(a_1) & g(a_2) \\ 0 & g(a_3) \end{pmatrix}$ for $g \in G$. Then

- (1) $A^G = Q$.
- (2) $B^G = \left\{ \begin{pmatrix} q_1 & q_2 \\ 0 & q_3 \end{pmatrix} \mid q_1, q_2, q_3 \in Q \right\}$, the ring of all 2-by-2 upper triangular matrices over Q .
- (3) $\Delta = V_B(B^G) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in A \right\} \cong A$.
- (4) Δ is a Galois extension of Δ^G with Galois group $G|_\Delta \cong G$ and a Galois system $\{1, i, j, k; 1/4, -i/4, -j/4, -k/4\}$.
- (5) $\Delta^G = Q$ is the center of Δ .
- (6) By (4) and (5), B is a central commutator Galois extension of B^G .
- (7) The center of B^G is Q .
- (8) B^G is not a separable extension of its center Q , and so B^G is not an Azumaya algebra. In fact, suppose that B^G is a separable extension of Q . Then, there exists a separable idempotent

$$e = \sum_{\substack{1 \leq i \leq j \leq 2 \\ 1 \leq k \leq l \leq 2}} q_{ijkl} (e_{ij} \otimes e_{kl}), \tag{4.2}$$

where $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $q_{ijkl} \in Q$ such that

$$\sum_{\substack{1 \leq i \leq j \leq 2 \\ 1 \leq k \leq l \leq 2}} q_{ijkl} e_{ij} e_{kl} = I_2, \tag{4.3}$$

the identity 2-by-2 matrix, and $be = eb$ for all $b \in B^G$. By $e_{11}e = ee_{11}$, we have

$$\sum_{\substack{1 \leq j \leq 2 \\ 1 \leq k \leq l \leq 2}} q_{1jkl} (e_{1j} \otimes e_{kl}) = \sum_{1 \leq i \leq j \leq 2} q_{ij11} (e_{ij} \otimes e_{11}). \tag{4.4}$$

Hence $q_{2211} = 0$ and $q_{1jk2} = 0$ for all j, k , that is, $q_{1112} = q_{1122} = q_{1212} = q_{1222} = 0$. By $e_{12}e = ee_{12}$, we have

$$\sum_{1 \leq k \leq l \leq 2} q_{22kl} (e_{12} \otimes e_{kl}) = \sum_{1 \leq i \leq j \leq 2} q_{ij11} (e_{ij} \otimes e_{12}). \tag{4.5}$$

Hence $q_{22kl} = 0$ if $(k, l) \neq (1, 2)$ and $q_{ij11} = 0$ if $(i, j) \neq (1, 2)$, that is, $q_{2211} = q_{2222} = 0$ and $q_{1111} = q_{2211} = 0$. Therefore, $e = q_{1211}(e_{12} \otimes e_{11}) + q_{2212}(e_{22} \otimes e_{12})$. Thus,

$$I_2 = \sum_{\substack{1 \leq i \leq j \leq 2 \\ 1 \leq k \leq l \leq 2}} q_{ijkl} e_{ij} e_{kl} = q_{1211} e_{12} e_{11} + q_{2212} e_{22} e_{12} = 0. \tag{4.6}$$

This contradiction shows that B^G is not a separable extension of Q .

EXAMPLE 4.4. Let $B = Q[i, j, k]$ be the quaternion algebra over the rational field Q and $G = \{1, g_i\}$ where $g_i(x) = ix i^{-1}$ for all x in B . Then

(1) B is a Galois extension of B^G with Galois group G and a Galois system $\{1, i, j, k; 1/4, -i/4, -j/4, -k/4\}$.

(2) Since G is inner, B is an H -separable extension of B^G .

(3) By (1) and (2), B is a Galois H -separable extension of B^G .

(4) $\Delta = V_B(B^G) = Q[i]$ is not a Galois extension of Δ^G with Galois group $G|_{\Delta} \cong G$, and so B is not a central commutator Galois extension of B^G .

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GEORGE SZETO: MATHEMATICS DEPARTMENT, BRADLEY UNIVERSITY, PEORIA, ILLINOIS 61625, USA

E-mail address: szeto@bradley.bradley.edu

LIANYONG XUE: MATHEMATICS DEPARTMENT, BRADLEY UNIVERSITY, PEORIA, ILLINOIS 61625, USA

E-mail address: lxue@bradley.bradley.edu



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