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AN EXOTIC CHARACTERIZATION OF A COMMUTATIVE H^*-ALGEBRA

PARFENY P. SAWOROTNOW

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Abstract. Commutative H^*-algebra is characterized in terms of the property that the orthogonal complement of a right ideal is a left ideal.

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1. Introduction. In recent years the author has been working on characterizing different classes of commutative Banach algebras (see [7, 8, 9]). In doing so he remembered a characterization result that he discovered a long time ago, but which he dismissed at that time as being both exotic and somewhat artificial. Now he feels that this characterization may be of interest to the mathematical community. In any case, it indicates importance of Jacobson's theory of radicals associated with a ring [3]; in particular it shows how strong is the assumption of semi-simplicity in the case of Hilbert algebras.

The present paper is devoted to this characterization.

2. Main results

Theorem 2.1. *Let A be a semi-simple complex Hilbert algebra (this means that A is a Banach algebra and it has an inner product $(\ ,\ )$ such that $(x,x) = \|x\|^2$ for all $x \in A$). Assume further that the orthogonal complement R^p of any right ideal R is a left ideal. Then A is a commutative H^*-algebra. In fact, A is a direct sum, $A = \sum_{\alpha\in\Gamma} I_\alpha$ of minimal two-sided ideals I_α, $\alpha \in \Gamma$, each of which is isomorphic to the complex field.*

To prove this theorem we shall need several lemmas.

Lemma 2.2. *The orthogonal complement R^p of a right ideal R is a two-sided ideal, it coincides with the right annihilator $r(R)$ of R.*

Proof. First note that R^p is included in the right annihilator $r(R)$ of R, since $RR^p \subset R\cap R^p = (0)$. Also $r(R)$ is a right ideal (in fact, it is a two-sided ideal): if $x \in r(R)$ and $a \in A$, then $y(xa) = (yx)a = 0$ for each $y \in R$ (and $y(ax) = (ya)x = 0$ since $ya \in R$). This means that the right ideal $R\cap r(R)$ annihilates itself (which means that $x^2 = 0$ if $x \in R\cap r(R)$). This shows that $R\cap r(R) = (0)$ (see [3]). Thus, $R^p = r(R)$. □

Corollary 2.3. *R^p is also the left annihilator of R, $R^p = l(R)$.*

Proof. Since R^p is also a right ideal, we may conclude that $R = r(R^p)$, which means that $R^p \subset l(R)$. Using the above argument we conclude that $R^p = l(R)$. $\square$

Corollary 2.4. *The algebra A is a right complemented algebra* [6].

Corollary 2.5. *Each closed right ideal in A is also a left ideal.*

Proof. Corollary 2.5 is a consequence of the fact that each right ideal R is the orthogonal complement of the right ideal R^p. $\square$

As it was defined in [6], a left projection is a non-zero member e of A which is both idempotent ($e^2 = e$) and left selfadjoint ($(ex, y) = (x, ey)$ for all $x, y \in A$). As in [1, 6] a left projection e is primitive if it cannot be represented as a sum, $e = e_1 + e_2$, of two left projections e_1 and e_2 such that $e_1e_2 = 0$ (which implies both "$e_2e_1 = 0$" and "$(e_1, e_2) = 0$").

Lemma 2.6. *Each closed non-zero right ideal R in A contains a primitive left projection.*

Proof. We can employ the proof of [6, Lemma 5] here (note that $l(R^p) = R$): take $x \in R$ which does not have a right quasi-inverse, consider the right ideal $R_1 =$ closure of $\{xy + x : y \in A\}$ and write $-x = e + u$ with $e \in R_1^p$, $u \in R_1$. Then e is a left projection in R (since $R_1^p \subset R$). If e is not primitive, then as in [1] one can show that e can be written as a finite sum, $e = \sum e_i$ of primitive left projections $e_1, \dots, e_n$ (note that $\|e\| \geq 1$ for each idempotent e), such that $e_ie_j = 0$ if $i \neq j$. Needless to say, each $e_i = ee_i$ is a member of R. $\square$

Lemma 2.7. *A left projection e is primitive if and only if the closed right ideal eA is minimal.*

Proof is left to the reader.

Lemma 2.8. *If e is a primitive left projection, then the right ideal $R = eA$ is a division algebra. In fact, eA is isomorphic to the complex field C, $eA = \{\lambda e : \lambda \in C\}$.*

Proof. First note that e is also a right identity of $eA(ae = a$ for each $a \in eA)$. It follows from the fact that R^p is a left annihilator of eA: if $a \in R$ then $ae - a \in l(eA) = R^p$ (but $ae - a$ is also a member of $R = eA$: thus, $ae - a = 0$, $ae = a$). $\square$

Now we show that each $x \in eA$ has both right and left inverses. As in the proof of [6, Lemma 6] one can show that the closed right ideal eA has no proper ideals (closed or not). It follows that each $x \in eA$ has a right inverse y (it is a consequence of the fact that $xA = eA$). But ey is also a right inverse of $x : xey = xy = e$. Also $ey \in eA$, and so it has a right inverse z, $eyz = e$. A standard argument shows that $x = ez : x = xe = xeyz = ez$, which implies that $eyx = e$, i.e., ey is also a left inverse of x. This proves that $R = eA$ is a division algebra. The last part follows from Gelfand-Mazur [4, Theorem 22F] (see also [2, Proposition 4.III in §9] and [5, Theorem 2 in §4]).

Lemma 2.9. *Product e_1e_2 of any two distinct primitive projections e_1, e_2 equals zero, $e_1e_2 = 0$.*

Proof. The ideals $R_1 = e_1A$ and $R_2 = e_2A$ are minimal, and $R_1 \cap R_2 \subset R_1, R_2$. This simply means that $R_1 \cap R_2 = (0)$, from which we conclude that $R_1R_2 = 0$ (since $R_1R_2 \subset R_1 \cap R_2$) (other possibility would be $R_1 \cap R_2 = R_i$ for $i = 1,2$, which is impossible, since $e_1 \neq e_2$). Thus $e_1e_2 = 0$. □

Corollary 2.10. *If e_1, e_2 are minimal projections then $e_1A \perp e_2A$ (which means that $(x,y) = 0$ for any $x \in e_1A$, $y \in e_2A$).*

Proof of the theorem. Let $\{e_\alpha : \alpha \in \Gamma\}$ be the family of all primitive projections in A. Then $A = \sum_{\alpha\in\Gamma} \alpha A$ (because of Lemma 2.6) and each $I_\alpha = e_\alpha A$ is isomorphic to the complex field. Thus A is commutative H^*-algebra since it is a direct sum of commutative H^*-algebras (the complex field is a one-dimensional H^*-algebra). □

Remark 2.11. In effect we characterized the algebra $L^2(S,k)$, described in the example below.

Example 2.12. Let S be a set (of any cardinality whatever), and let $k(s)$ be a real valued function on S such that $k(s) \geq 1$ for all $s \in S$. Let $L^2(S,k)$ be the algebra of all complex valued functions $x(s)$ on S such that $\sum_{s\in S} |x(s)|^2 k(s) < \infty$ (this means there exists a countable subset S_x of S such that $k(s) = 0$ if $s \in S \sim S_x$ and the series $\sum_{s\in S_x} |x(s)|^2 k(s)$ converges).

Theorem 2.13. *The algebra $L^2(S,k)$ is a commutative H^*-algebra with respect to the point wise addition and multiplications, the scalar product $(x,x') = \sum_{s\in S} x(s)\bar{x}'(s)$ $k(s)$ and the involution $x \to x^*$ defined by $x^*(s) = \bar{x}(s)$. Conversely, for each commutative proper H^*-algebra A there exist a set S and a real valued function $k(s)$ with $k(s) \geq 1$ such that A is isomorphic and isometric to $L^2(S,k)$.*

Proof. First part is established by direct verification, the second part was in effect established in [1]. It is an easy consequence of Lemma 2.8 above (note that each proper commutative H^*-algebra A satisfies assumption of Theorem 2.1 above): all we have to do is to take S to be the set of all minimal projections e in A and define the function $k()$ by setting $k(e) = \|e\|$. □

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Parfeny P. Saworotnow: Department of Mathematics, Catholic University of America, Washington, DC 20064, USA
E-mail address: `saworotnow@cua.edu`


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