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A UNIFIED THEORY FOR WEAK SEPARATION PROPERTIES

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ABSTRACT. We devise a framework which leads to the formulation of a unified theory of normality (regularity), semi-normality (semi-regularity), s-normality (s-regularity), feebly-normality (feebly-regularity), pre-normality (pre-regularity), and others. Certain aspects of theory are given by unified proof.

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1. Introduction. Normality and regularity are some of the important properties for studying topological spaces. Several types of normality and regularity occur in the literature. In this paper, we apply the concept of φ-operation which was defined by Csaszar [7] to unify and generalize several characterizations and properties of a lot of already existing weaker forms of normality and regularity. In Section 2, we introduce the concept of $\varphi\psi$-normality and $\varphi\psi$-regularity as a generalization of the concepts of normality, regularity, semi-normality [17], semi-regularity [14], s-normality [23], s-regularity [21], feebly-normality [18], feebly-regularity [18], β-normality [25], β-regularity [2], and p-regularity [19]. Also, we introduce the concepts of φ-R_0 and φ-T_0-identification spaces as generalizations of the concepts of R_0-space [10], T_0-identification space [28], semi-R_0-space [22], semi-T_0-identification space [14], and s-feebly-R_0-space [16] to unify and generalize several characterizations. In Section 3, characterizations and properties of $\varphi\psi$-normality and $\varphi\psi$-regularity are investigated.

Throughout the paper, (X,τ) and (Y,ϑ) mean topological spaces on which no separation axioms are assumed unless otherwise explicitly stated. Let A be a subset of X, A is called semi-open [20] (respectively, pre-open [26], feebly open [24], and β-open [1]) if $A \subseteq \operatorname{cl}(\operatorname{int} A)$ (respectively, $A \subseteq \operatorname{int}(\operatorname{cl} A)$, $A \subseteq \operatorname{scl}(\operatorname{int} A)$, and $A \subseteq \operatorname{cl}(\operatorname{int}(\operatorname{cl} A))$. Semi-$T_i$ [22] (respectively, pre-T_i, feebly-T_i [18], and β-T_i [25]) spaces are defined as ordinary ones except each open set is replaced by semi-open (respectively, pre-open, feebly open, and β-open) one, $i = 0,1$. A space (X,τ) is R_0 [10] if and only if for each $O \in \tau$ and $x \in O$, $\operatorname{cl}\{x\} \subseteq O$. (X,τ) is semicompact [12] if for every semi-open cover of it has a finite subcover. A space (X,τ) is extremally disconnected (E.D.) [5, 6] if $\operatorname{cl} O \in \tau$ for each $O \in \tau$ and is submaximal [5, 6] if all dense sets in it are open. A space (X,τ) is s-weakly Hausdorff [11] if for each pair $x,y \in X$ such that $\operatorname{cl}\{x\} \neq \operatorname{cl}\{y\}$, there exist disjoint semi-open sets U and V such that $x \in U$ and $y \in V$. A space (X,τ) is R_1 [8] (semi-R_1 [9]) if and only if for each pair $x,y \in X$ such that $\operatorname{cl}\{x\} \neq \operatorname{cl}\{y\}$ ($\operatorname{scl}\{x\} \neq \operatorname{scl}\{y\}$), there exist disjoint open (semi-open) sets U and V such that $\operatorname{cl}\{x\} \subseteq U$ and $\operatorname{cl}\{y\} \subseteq V$ ($\operatorname{scl}\{x\} \subseteq U$ and $\operatorname{scl}\{y\} \subseteq V$).

2. A unified framework

Definition 2.1 (see [7]). Let (X,τ) be a topological space. A mapping $\varphi : P(X) \to P(X)$ is called an operation on $P(X)$, where $P(X)$ denotes the family all of the subsets of X if and only if for each $A \in P(X) - \{\emptyset\}$, $\operatorname{int} A \subseteq A^{\varphi}$ and $\emptyset = \emptyset^{\varphi}$, where A^{φ} is denotes the value of φ in A. The class of all operations on $P(X)$ is denoted by $O(X)$.

Throughout the paper, all of the operations on $P(X)$ are assumed to be monotonous (i.e., such that $A \subseteq B$ implies $A^{\varphi} \subseteq B^{\varphi}$).

Definition 2.2 (see [7]). Let (X,τ) be a topological space, $G,H \in P(X)$, and $\varphi \in O(X)$. Then

(i) G is called φ-open if and only if $G \subseteq G^{\varphi}$.

(ii) The subset H is called φ-closed if and only if $X - H$ is φ-open.

The class of all φ-open (φ-closed) subsets of X is denoted by $\varphi O(X)(\varphi C(X))$. For each $x \in X$, the set $\{V \subseteq X \mid x \in V \in \varphi O(X)\}$ is denoted by $N(x,\varphi O(X))$.

Definition 2.3 (see [7]). Let (X,τ) be a topological space, $G \in P(X)$, and $\varphi \in O(X)$.

(i) The intersection of all φ-closed sets containing G is the φ-closure of G, denoted by $\varphi\operatorname{cl} G$.

(ii) The union of all φ-open subsets of G is the φ-interior of G, denoted by $\varphi\operatorname{int} G$.

The set $\varphi\operatorname{cl} G$ is the smallest φ-closed set containing G, and the set $\varphi\operatorname{int} G$ is the largest φ-open subset of G.

In a topological space, if $\varphi = \operatorname{int}$, then $\varphi\operatorname{cl} = \operatorname{cl}$. Similarly, if $\varphi = \operatorname{int} \circ \operatorname{cl}$, then $\varphi\operatorname{cl} = \operatorname{pcl}$, where pcl denotes pre-closure.

The following lemma is obvious above definitions.

Lemma 2.4. *Let (X,τ) be a topological space, $G \in P(X)$, $\varphi \in O(X)$, and $x \in X$.*

(i) $x \in \varphi\operatorname{cl} G \Leftrightarrow \forall T \in N(x,\varphi O(X))(G \cap T \neq \emptyset)$.

(ii) $x \in \varphi\operatorname{int} G \Leftrightarrow \exists T \in N(x,\varphi O(X))(x \in T \subseteq G)$.

Definition 2.5. A topological space (X,τ) is

(i) a $\varphi - T_0$-space if, for any two distinct points x and y of X, there exists either a φ-open set containing x but not y or a φ-open set containing y but not x.

(ii) a $\varphi - T_1$-space if, for any two distinct points x and y of X, there exists a φ-open set containing x but not y and a φ-open set containing y but not x.

Definition 2.6. Let (X,τ) be a topological space, and $\varphi,\psi \in O(X)$.

(i) (X,τ) is called $\varphi\psi$-normal if and only if for each pair of disjoint φ-closed sets A and B, there exist disjoint ψ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(ii) (X,τ) is called ψ-normal if and only if $\varphi = \psi$.

Definition 2.7. Let (X,τ) be a topological space, and $\varphi,\psi \in O(X)$.

(i) (X,τ) is called $\varphi\psi$-regular if and only if for each φ-closed set A and $x \notin A$, there exist disjoint ψ-open sets U and V such that $x \in U$ and $A \subseteq V$.

(ii) (X,τ) is called ψ-regular if and only if $\varphi = \psi$.

Definition 2.8. Let (X,τ) be a topological space and $\varphi \in O(X)$. (X,τ) is called $\varphi - R_0$ if and only if for each $O \in \varphi O(X)$ and $x \in O$, $\varphi\operatorname{cl}\{x\} \subseteq O$.

TABLE 2.1.

φ-ope.	ψ-ope.	$\varphi\psi$-normal space	$\varphi\psi$-regular space	$\varphi - R_0$ space	$\varphi - T_0$ identification space	$\psi - T_0$ identification space
int	int	normal	regular	R_0 [10]	T_0-iden. [28]	T_0-iden. [28]
$\mathrm{cl} \circ \mathrm{int}$	$\mathrm{cl} \circ \mathrm{int}$	semi-normal [17]	semi-regular [14]	semi-R_0 [22]	semi-T_0-iden. [13]	semi-T_0-iden. [13]
$\mathrm{int} \circ \mathrm{cl}$	$\mathrm{int} \circ \mathrm{cl}$	pre-normal [27]	pre-regular	s-pre-R_0	pre-T_0-iden.	pre-T_0-iden.
$\mathrm{scl} \circ \mathrm{int}$	$\mathrm{scl} \circ \mathrm{int}$	s-feebly-normal	s-feebly-regular	s-feebly-R_0 [16]	feebly-T_0-iden.	feebly-T_0-iden.
$\mathrm{cl} \circ \mathrm{int} \circ \mathrm{cl}$	$\mathrm{cl} \circ \mathrm{int} \circ \mathrm{cl}$	s-β-normal	s-β-regular	s-$\beta - R_0$	$\beta - T_0$.-iden.	$\beta - T_0$.-iden.
int	$\mathrm{cl} \circ \mathrm{int}$	s-normal [23]	s-regular [21]	R_0 [10]	T_0-iden. [28]	semi-T_0-iden. [13]
int	$\mathrm{int} \circ \mathrm{cl}$	p-normal	p-regular [19]	R_0 [10]	T_0-iden. [28]	pre-T_0-iden.
int	$\mathrm{scl} \circ \mathrm{int}$	feebly-normal [18]	feebly-regular [18]	R_0 [10]	T_0-iden. [28]	feebly-T_0-iden.
int	$\mathrm{cl} \circ \mathrm{int} \circ \mathrm{cl}$	β-normal [25]	β-regular [2]	R_0 [10]	T_0-iden. [28]	$\beta - T_0$-iden.

DEFINITION 2.9. Let (X,τ) be a topological space and $\varphi \in O(X)$. Let R be the equivalence relation on the space (X,τ) defined by xRy if and only if $\varphi\,\mathrm{cl}\{x\} = \varphi\,\mathrm{cl}\{y\}$. The $\varphi - T_0$-identification space of (X,τ) is $(X_\varphi, Q(X_\varphi))$, where X_φ is the set of equivalence classes of R and $Q(X_\varphi)$ is the decomposition topology on X_φ. Let $P_\varphi : (X,\tau) \to (X_\varphi, Q(X_\varphi))$ denote the natural map.

Table 2.1 lists the type of $\varphi\psi$-normal, $\varphi\psi$-regular, $\varphi - R_0$, and φ (ψ)-T_0-identification spaces induced by operations φ and ψ, and gives explicitly the weak forms of normality and regularity to which they correspond. Besides, some new definitions appear in the developing of the unified theory.

3. Characterizations and properties of $\varphi\psi$-normality and $\varphi\psi$-regularity

THEOREM 3.1. *A space (X,τ) is $\varphi\psi$-normal if and only if for each φ-closed set A and φ-open set U containing A, there exists a ψ-open set G such that $A \subseteq G \subseteq \psi\,\mathrm{cl}\,G \subseteq U$.*

PROOF. $(\Rightarrow)$ Let A be a φ-closed set and U be a φ-open set containing A. Then $A \cap (X - U) = \emptyset$ and $X - U$ is a φ-closed set. There exist ψ-open sets G and H such that $A \subseteq G$ and $X - U \subseteq H$. Therefore, we have $A \subseteq G \subseteq X - H \subseteq U$ and hence $\psi\,\mathrm{cl}\,G \subseteq \psi\,\mathrm{cl}(X - H) \subseteq U$ since $X - H$ is ψ-closed. Consequently, we obtain $A \subseteq G \subseteq \psi\,\mathrm{cl}\,G \subseteq U$.

$(\Leftarrow)$ Let A and B be disjoint φ-closed sets. Then $A \subseteq X - B$ and $X - B$ is a φ-open set. There exists a ψ-open set G such that $A \subseteq G \subseteq \psi\,\mathrm{cl}\,G \subseteq X - B$. Therefore, we have $A \subseteq G$, $B \subseteq X - \psi\,\mathrm{cl}\,G$ and $G \cap (X - \psi\,\mathrm{cl}\,G) = \emptyset$. This shows that X is $\varphi\psi$-normal. $\square$

Theorem 3.2. *The following properties are equivalent for a space X:*

(1) *X is $\varphi\psi$-regular;*

(2) *for each $x \in X$ and $U \in \varphi O(X)$ such that $x \in U$, there exists a ψ-open set V such that $x \in V \subseteq \psi\,\mathrm{cl}\,V \subseteq U$;*

(3) *for each φ-closed set F of X, $F = \cap\{\psi\,\mathrm{cl}\,W \mid F \subseteq W,\ W \in \psi O(X)\}$;*

(4) *for each subset $A \subseteq X$ and each φ-open set U such that $A \cap U \neq \emptyset$, there exists $W \in \psi O(X)$ such that $A \cap W \neq \emptyset$ and $\psi\,\mathrm{cl}\,W \subseteq U$;*

(5) *for each nonempty subset $A \subseteq X$ and each φ-closed set F of X such that $A \cap F = \emptyset$, there exist $U, W \in \psi O(X)$ such that $A \cap U \neq \emptyset$, $F \subseteq W$, and $U \cap W = \emptyset$.*

Proof. (1)⇒(2). Let U be a φ-open set containing x, then $X - U$ is φ-closed set in X and $x \notin X - U$. By (1), there are two disjoint ψ-open sets V, H such that $x \in V$ and $X - U \subseteq H$. Therefore, we have $x \in V \subseteq X - H \subseteq U$ and hence $x \in V \subseteq \psi\,\mathrm{cl}\,V \subseteq U$.

(2)⇒(3). Let F be a φ-closed set of X. It is obvious that $F \subseteq \cap\{\psi\,\mathrm{cl}\,W \mid F \subseteq W, W \in \psi O(X)\}$. Conversely, let $x \notin F$ and $F \in \varphi C(X)$. Then $X - F$ is φ-open set containing x, by (2) there is a ψ-open set V such that $x \in V \subseteq \psi\,\mathrm{cl}\,V \subseteq X - F$. Put $W = X - \psi\,\mathrm{cl}\,V$, it follows that $F \subseteq W \in \psi O(X)$ and $x \notin \psi\,\mathrm{cl}\,W$. This implies that $\cap\{\psi\,\mathrm{cl}\,W \mid F \subseteq W, W \in \psi O(X)\} \subseteq F$.

(3)⇒(4). Let U be φ-open set in X and $A \cap U \neq \emptyset$. For $x \in A \cap U$, $X - U$ is φ-closed set not containing x, by (3) there is a $V \in \psi O(X)$ such that $X - U \subseteq V$ and $x \notin \psi\,\mathrm{cl}\,V$. Put $W = X - \psi\,\mathrm{cl}\,V$, we obtain $W \in \psi O(X)$, $x \in W \cap A \neq \emptyset$ and $\psi\,\mathrm{cl}\,W \subseteq \psi\,\mathrm{cl}(X - V) = X - V \subseteq U$.

(4)⇒(5). Let F be φ-closed set in X and $A \cap F = \emptyset$, where $A \neq \emptyset$. Since $X - F$ is φ-open in X and A is nonempty, by (4) there is a $W \in \psi O(X)$ such that $A \cap W \neq \emptyset$ and $\psi\,\mathrm{cl}\,W \subseteq X - F$. Put $V = X - \psi\,\mathrm{cl}\,W$. Then $F \subseteq V \in \psi O(X)$ and $W \cap V = \emptyset$.

(5)⇒(1). It is obvious. □

Theorem 3.3. *If (X, τ) is $\varphi\psi$-normal and $\varphi - R_0$, then (X, τ) is $\varphi\psi$-regular.*

Proof. Let A be a φ-closed set not containing x. Then $x \in X - A$ is φ-open set, which implies $\varphi\,\mathrm{cl}\{x\} \subseteq X - A$, and there exist disjoint ψ-open sets U and V such that $x \in \varphi\,\mathrm{cl}\{x\} \subseteq U$ and $A \subseteq V$.

Referring to Table 2.1, Theorem 3.3 contains several results in the literature. For example, with $\varphi = \psi = \mathrm{cl} \circ \mathrm{int}$ it gives that if (X, τ) is semi-normal, semi-R_0, then (X, τ) is semi-regular, a result due to Dorsett [17]. Similarly, with $\varphi = \mathrm{int}$, $\psi = \mathrm{cl} \circ \mathrm{int}$ it yields a result due to Maheshwari and Prasad [23]. □

Example 3.12 [14] shows that the converse of Theorem 3.3 is false.

Theorem 3.4. *The natural map $P_\varphi : (X, \tau) \to (X_\varphi, Q(X_\varphi))$ is closed, open, and $P_\varphi^{-1}(P_\varphi(O)) = O$ for all $O \in \varphi O(X)$ and $(X_\varphi, Q(X_\varphi))$ is $\varphi - T_0$.*

Proof. For each $x \in X$, let C_x be the equivalence class of R containing x. Let $O \in \varphi O(X)$ and $x \in P_\varphi^{-1}(P_\varphi(O))$. Then there exists $y \in O$ such that $C_x = C_y$. Thus $\varphi\,\mathrm{cl}\{x\} = \varphi\,\mathrm{cl}\{y\}$ and $O \in \varphi O(X)$ such that $y \in O$, which implies $x \in O$. Hence $P_\varphi^{-1}(P_\varphi(O)) \subseteq O$, which implies $O = P_\varphi^{-1}(P_\varphi(O))$. Since $\tau \subseteq \varphi O(X)$, then $P_\varphi^{-1}(P_\varphi(U)) = U$ for all $U \in \tau$, which implies P_φ is closed and open.

Let $A, B \in X_\varphi$ such that $A \neq B$ and let $x \in A$ and $y \in B$. Then $\varphi\,\mathrm{cl}\{x\} \neq \varphi\,\mathrm{cl}\{y\}$,

which implies $x \notin \varphi\,\mathrm{cl}\{y\}$ or $y \notin \varphi\,\mathrm{cl}\{x\}$, say, $x \notin \varphi\,\mathrm{cl}\{y\}$. Since P_φ is continuous and open, then $A \in D = P_\varphi(X - \varphi\,\mathrm{cl}\{y\}) \in \varphi O(X_\varphi, Q(X_\varphi))$ and $B \notin D$.

Using Table 2.1, Theorem 3.4 unifies several known results. For example, if $\varphi = \mathrm{int}$, then it yields Theorem 2.1 of Dorsett [10]. Similarly, with $\varphi = \mathrm{cl} \circ \mathrm{int}$ it yields Theorem 2.1 of Dorsett [13]. □

Theorem 3.5. *The following are equivalent:*

(a) (X,τ) *is* $\varphi - R_0$,

(b) $X_\varphi = \{\varphi\,\mathrm{cl}\{x\} \mid x \in X\}$,

(c) $(X_\varphi, Q(X_\varphi))$ *is* $\varphi - T_1$.

Proof. (a)⇒(b). Let $C \in X_\varphi$ and $x \in C$. If $y \in C$, then $y \in \varphi\,\mathrm{cl}\{y\} = \varphi\,\mathrm{cl}\{x\}$, which implies $C \subseteq \varphi\,\mathrm{cl}\{x\}$. If $y \in \varphi\,\mathrm{cl}\{x\}$, then $x \in \varphi\,\mathrm{cl}\{y\}$. Since otherwise, $x \in X - \varphi\,\mathrm{cl}\{y\} \in \varphi O(X)$, which implies $\varphi\,\mathrm{cl}\{x\} \subseteq X - \varphi\,\mathrm{cl}\{y\}$, which is a contradiction. Thus, if $y \in \varphi\,\mathrm{cl}\{x\}$, then $x \in \varphi\,\mathrm{cl}\{y\}$, which implies $\varphi\,\mathrm{cl}\{y\} = \varphi\,\mathrm{cl}\{x\}$ and $y \in C$.

(b)⇒(c). Let $A, B \in X_\varphi$ such that $A \neq B$. Then there exist $x, y \in X$ such that $A = \varphi\,\mathrm{cl}\{x\}$ and $B = \varphi\,\mathrm{cl}\{y\}$, and $\varphi\,\mathrm{cl}\{x\} \cap \varphi\,\mathrm{cl}\{y\} = \emptyset$. Then $A \in C = P_\varphi(X - \varphi\,\mathrm{cl}\{y\}) \in \varphi O(X_\varphi, Q(X_\varphi))$ and $B \notin C$.

(c)⇒(a). Let $O \in \varphi O(X,\tau)$ and let $x \in O$. Let $y \notin O$ and let $C_x, C_y \in X_\varphi$ containing x, y, respectively. Then $x \notin \varphi\,\mathrm{cl}\{y\}$, which implies $C_x \neq C_y$ and there exists a φ-open set A such that $C_y \in A$ and $C_x \notin A$. Since P_φ is continuous and open, then $y \in B = P_\varphi^{-1}(A) \in \varphi O(X,\tau)$ and $x \notin B$, which implies $y \notin \varphi\,\mathrm{cl}\{x\}$. Thus $\varphi\,\mathrm{cl}\{x\} \subseteq O$.

According to Table 2.1, Theorem 3.5 represents the unification of various results in the literature. For example, if $\varphi = \mathrm{cl} \circ \mathrm{int}$, we get a result pertaining to semi-R_0 space. Thus, the following are equivalent: (a) (X,τ) is semi-R_0, (b) $X_s = \{\mathrm{scl}\{x\} \mid x \in X\}$, (c) $(X_s, Q(X_s))$ is semi-T_1 [13, Theorem 2.2]. The classical result pertaining to R_0 space follows by substituting int for φ. □

Theorem 3.6. *A space* (X,τ) *is* $\varphi\psi$*-normal if and only if its* $\psi - T_0$*-identification space* $(X_\psi, Q(X_\psi))$ *is* $\varphi\psi$*-normal, where* $\varphi \leq \psi$.

Proof. (⇒) Let A and B be disjoint φ-closed sets in X_ψ. Since P_ψ is continuous, open and $\varphi \leq \psi$, $P_\psi^{-1}(A)$ and $P_\psi^{-1}(B)$ are disjoint φ-closed sets in X. Then there exist disjoint ψ-open sets U and V such that $P_\psi^{-1}(A) \subseteq U$ and $P_\psi^{-1}(B) \subseteq V$. Since P_ψ is continuous, open and $P_\psi^{-1}(P_\psi(U)) = U$ for all $U \in \psi O(X)$, then $P_\psi(U)$ and $P_\psi(V)$ are disjoint ψ-open sets in X_ψ such that $A \subseteq P_\psi(U)$ and $B \subseteq P_\psi(V)$.

(⇐) Let A and B be disjoint φ-closed sets in X. Since P_ψ is continuous, open and $P_\psi^{-1}(P_\psi(C)) = C$ for all $C \in \psi O(X)$, $P_\psi(A)$ and $P_\psi(B)$ are disjoint φ-closed sets in X_ψ. Then there exist disjoint ψ-open sets U and V in X_ψ such that $P_\psi(A) \subseteq U$ and $P_\psi(B) \subseteq V$. Since P_ψ is continuous and open, then $P_\psi^{-1}(U)$ and $P_\psi^{-1}(V)$ are disjoint ψ-open sets in X containing A and B, respectively.

Referring to Table 2.1, Theorem 3.6 contains several results in the literature. For example, with $\varphi = \psi = \mathrm{cl} \circ \mathrm{int}$ it gives that a space (X,τ) is semi-normal if and only if its semi-T_0-identification space $(X_s, Q(X_s))$ is semi-normal, a result due to Dorsett [17]. Similarly, with $\varphi = \mathrm{int}$, $\psi = \mathrm{cl} \circ \mathrm{int}$ it gives that a space (X,τ) is s-normal if and only if its *semi*-T_0-identification space $(X_s, Q(X_s))$ is s-normal, a result due to Dorsett [15]. □

Theorem 3.7. *A space (X,τ) is $\varphi\psi$-regular if and only if its $\psi - T_0$-identification space $(X_\psi, Q(X_\psi))$ is $\varphi\psi$-regular, where $\varphi \le \psi$.*

Proof. ($\Rightarrow$) Let A be φ-closed set in X_ψ and let $C_x \notin A$, where C_x is the equivalence class containing x. Since P_ψ is continuous and open, then $P_\psi^{-1}(A)$ is φ-closed set in X not containing x. Then there exist disjoint ψ-open sets U and V such that $x \in U$ and $P_\psi^{-1}(A) \subseteq V$. Since P_ψ is continuous, open and $P_\psi^{-1}(P_\psi(U)) = U$ for all $U \in \psi O(X)$, then $P_\psi(U)$ and $P_\psi(V)$ are disjoint ψ-open sets in X_ψ such that $P_\psi(x) = C_x \in P_\psi(U)$ and $A \subseteq P_\psi(V)$.

($\Leftarrow$) Let A be disjoint φ-closed set in X not containing x. Since P_ψ is continuous, open and $P_\psi^{-1}(P_\psi(A)) = A$ for all $A \in \psi O(X)$, then $P_\psi(A)$ is φ-closed set in X_ψ and $C_x \notin P_\psi(A)$. Then there exist disjoint ψ-open sets U and V in X_ψ such that $C_x \in U$ and $P_\psi(A) \subseteq V$ and $P_\psi^{-1}(U)$ and $P_\psi^{-1}(V)$ are disjoint ψ-open sets in X containing x and A, respectively. □

Using Table 2.1, Theorem 3.7 unifies several known results. For example, if $\varphi = \psi = \mathrm{cl} \circ \mathrm{int}$, then it yields Theorem 2.1((a)$\Leftrightarrow$(b)) of Dorsett [14]. Similarly, with $\varphi = \mathrm{int}$, $\psi = \mathrm{cl} \circ \mathrm{int}$ it yields Theorem 2.1 of Dorsett [15].

Theorem 3.8. *A space (X,τ) is ψ-normal, then (X,τ) is $\varphi\psi$-normal, where $\varphi \le \psi$.*

Proof. Let A and B be disjoint φ-closed sets in X. Since $\varphi \le \psi$, A and B are disjoint ψ-closed sets and then there exist disjoint ψ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Referring to Table 2.1, Theorem 3.8 contains several results in the literature. For example, with $\varphi = \mathrm{int}$, $\psi = \mathrm{cl} \circ \mathrm{int}$ it gives that if a space (X,τ) is semi-normal, then (X,τ) is s-normal, a result due to Dorsett [17]. □

Example 2.1 [17] shows that converse of the Theorem 3.8 is false.

Theorem 3.9. *A space (X,τ) is ψ-regular, then (X,τ) is $\varphi\psi$-regular, where $\varphi \le \psi$.*

Proof. The proof is similar to that of Theorem 3.8. □

Referring to Table 2.1, Theorem 3.9 contains several results in the literature. For example, with $\varphi = \mathrm{int}$, $\psi = \mathrm{cl} \circ \mathrm{int}$ it gives that if a space (X,τ) is semi-regular, then (X,τ) is s-regular, a result due to Dorsett [14].

Example 3.2 [11] shows that converse of the Theorem 3.9 is false.

Lemma 3.10. *If (X,τ) is semicompact R_0, then (X,τ) is E.D.*

Proof. Let A be an open set in X. Since every open set is also closed [17, Theorem 2.4], $A \subseteq \mathrm{int}(\mathrm{cl}\,A)$ and $\mathrm{cl}\,A \subseteq \mathrm{int}(\mathrm{cl}\,A)$. Therefore, we obtain $\mathrm{cl}\,A$ is open set in X. □

In [4], it was shown that for a subset A of (X,τ), $\mathrm{int}\,A \subseteq \alpha\mathrm{int}\,A \subseteq p\,\mathrm{int}\,A \subseteq \beta\mathrm{int}\,A$, $\mathrm{int}\,A \subseteq \alpha\mathrm{int}\,A \subseteq s\,\mathrm{int}\,A \subseteq \beta\mathrm{int}\,A$, and in [16], $\mathrm{int}\,A \subseteq f\,\mathrm{int}\,A \subseteq s\,\mathrm{int}\,A \subseteq \beta\mathrm{int}\,A$, where $\alpha\mathrm{int}\,A$ (respectively, $f\,\mathrm{int}\,A$, $s\,\mathrm{int}\,A$, $p\,\mathrm{int}\,A$, and $\beta\mathrm{int}\,A$) is α-interior (respectively, feebly-interior, semi-interior, pre-interior, and β-interior) of A. Later in [3], it was shown that if (X,τ) is submaximal and E.D., then $\tau = \beta O(X) = \{A \subseteq X \mid A \text{ is } \beta\text{-open set}\}$. Therefore, by Lemma 3.10, if (X,τ) is semicompact R_0 and submaximal, $\tau = \beta O(X) = \{A \subseteq X \mid A \text{ is } \beta\text{-open set}\}$. Consequently, we obtain the following.

Corollary 3.11. *If (X,τ) is semicompact R_0 and submaximal, then all of the $\varphi\psi$-normality and $\varphi\psi$-regularity for the (X,τ) are equivalent.*

The next result follows immediately from Theorem 2.4 [17] and Corollary 3.11.

Corollary 3.12. *If (X,τ) is semicompact R_0 and submaximal, then the following are equivalent:*

(1) *(X,τ) is $\varphi\psi$-normal,*
(2) *(X,τ) is $\varphi\psi$-regular,*
(3) *$\{\varphi\,\mathrm{cl}\{x\} \mid x \in X\}$ is finite,*
(4) *(X,τ) is s-weakly Hausdorff,*
(5) *(X,τ) is semi-R_1,*
(6) *(X,τ) is R_1,*
(7) *(X,τ) is completely regular.*

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