Internat. J. Math. & Math. Sci.
Vol. 23, No. 6 (2000) 431–434
S0161171200002210

AN INTERESTING FAMILY OF CURVES OF GENUS 1

ANDREW BREMNER

(Received 22 October 1998)

ABSTRACT. We study the family of elliptic curves $y^2 = x^3 - t^2x + 1$, both over $\mathbb{Q}(t)$ and over $\mathbb{Q}$. In the former case, all integral solutions are determined; in the latter case, computation in the range $1 \le t \le 999$ shows large ranks are common, giving a particularly simple example of curves which (admittedly over a small range) apparently contradict the once held belief that the rank under specialization will tend to have minimal rank consistent with the parity predicted by the Selmer conjecture.

Keywords and phrases. Elliptic curve, integral solution, rank.

2000 Mathematics Subject Classification. Primary 11G05.

1. It is the purpose of this note to draw attention to the family of elliptic curves

$$E : y^2 = x^3 - t^2x + 1 \tag{1.1}$$

which tend to possess many integral points for a fixed $t \in \mathbb{Z}$; for example, at least 52 integral points when $t = 8$. We describe the computational investigation into this phenomenon. First, regarding (1.1) as the generic fibre of an elliptic surface defined over $\mathbb{C}$, then the group G of sections of the surface is naturally identified with the group of $\mathbb{C}(t)$-rational solutions of (1.1), which by abuse of notation will also be denoted by G. Indeed, we tend to refer to (1.1) as an elliptic curve over $\mathbb{C}(t)$ and speak about "points on the curve" rather than "sections of the surface". It is well known that G is finitely generated. For (1.1), the $\mathbb{C}(t)$-rank is 4, and this allows the determination of all the "integer" points of (1.1), which we effect over $\mathbb{Q}$, that is, finding all points (x, y) of (1.1) satisfying $x, y \in \mathbb{Q}[t]$.

Second, we give numerical information about the curves (1.1) in the range $1 \le t \le 999$, computing the rational Mordell-Weil rank independently of any of the standard conjectures, except in four instances where it was necessary to assume the Birch and Swinnerton-Dyer conjecture. The ranks are surprisingly large: although the $\mathbb{Q}(t)$-rank of (1.1) is 3, there are 9 cases of $\mathbb{Q}$-rank 7 and 82 cases of $\mathbb{Q}$-rank 6 within the given range. This provides some support for the belief that a curve under specialization will not necessarily tend to have the minimal rank consistent with the parity predicted by the Selmer conjecture.

2. The curve (1.1) occurs as Example 2.32(b) Cox and Zucker [3] where it is shown that the curve is torsion-free, and G has $\mathbb{C}(t)$-rank 4, with basis

$$P_0 = (0, 1), \qquad P_1 = (t, 1), \qquad P_2 = (-1, t), \qquad P_3 = (-w, w^2t) \tag{2.1}$$

where $w^2 + w + 1 = 0$.

Alternatively, this is a straightforward verification using the theory of Mordell-Weil lattices as described by Shioda [7]; see Kuwata and Top [5] for a typical illustration of the method.

Let $\langle\ ,\ \rangle : G \times G \to \mathbb{R}$ be the height-pairing defined by

$$\langle Q, Q' \rangle = \frac{1}{2}(\hat{h}(Q+Q') - \hat{h}(Q) - \hat{h}(Q')), \tag{2.2}$$

where $\hat{h}$ is the canonical height function. Using the algorithm of Silverman [8], one computes the height-pairing matrix on the basis (2.1) to be

$$(\langle P_i, P_j \rangle) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}. \tag{2.3}$$

Accordingly,

$$\hat{h}(m_0P_0 + m_1P_1 + m_2P_2 + m_3P_3) = \frac{1}{2}(m_0^2 - m_0m_1 + m_1^2 + m_1(m_2+m_3) + m_2^2 + m_3^2). \tag{2.4}$$

It is now, in theory at least, straightforward to determine all the points on (1.1) with $x, y \in \mathbb{C}[t]$. Hindry and Silverman [4, Corollary 8.5], give an explicit bound for the height of such points P, which for the curves (1.1) becomes $\hat{h}(P) \le 29$. Using (2.4), the coefficients m_0, m_1, m_2, m_3 of P with respect to the basis (2.1) must therefore satisfy

$$2m_0^2 + 2(m_0 - m_1)^2 + (m_1 + 2m_2)^2 + (m_1 + 2m_3)^2 \le 8 \times 29 = 232. \tag{2.5}$$

There are in fact 16873 four-tuples (m_0, m_1, m_2, m_3) satisfying (2.5), where we restrict without loss of generality to the first nonzero m_i being positive. The labour required to check in each of these cases whether the corresponding P has coefficients in $\mathbb{C}[t]$ is excessive, and we restrict the computation to $\mathbb{Q}[t]$. Adding P_3 to its conjugate point $(-w^2, wt)$ results in $P_0 + 2P_1 - P_2$ and it follows that a $\mathbb{Q}(t)$-basis of (1.1) is simply $\{P_0, P_1, P_2\}$. There are now 1330 solutions of (2.5) (with $m_3 = 0$) and it is practicable to test these for integrability using a program in PARI-gp. (The "largest" triple to test is $(m_0, m_1, m_2) = (7, 10, -7)$; the program ran for several weeks at low priority on a SUN workstation). The following result is established.

Theorem 2.1. *The curve (1.1) has precisely 24 solutions in $\mathbb{Q}[t]$, namely:*

$$\begin{aligned} (x, \pm y) = {} & (0,1), (-1,t), (t,1), (-t,1), (t+2, 2t+3), (-t+2, -2t+3), \\ & (t^2-1, t^3-2t), (t^2+2t+2, t^3+3t^2+4t+3), \\ & (t^2-2t+2, -t^3+3t^2-4t+3), \left(\frac{t^4}{4}, \frac{t^6}{8} - 1\right), \\ & (t^4+2t, t^6+3t^3+1), (t^4-2t, t^6-3t^3+1). \end{aligned} \tag{2.6}$$

3. Using Cremona's package "mrank" on a SUN workstation, we computed the rational rank of (1.1) when specialized to t in the range $1 \le t \le 999$. There were just four instances where the rank was not calculated exactly, corresponding to the presence of a nontrivial 2-component of the Safarevic-Tate group. Here Connell's "Apecs" package was used to compute the appropriate derivative of the L-series, thereby establishing the rank subject to the Birch and Swinnerton-Dyer conjecture. The tabulation of ranks is as follows: what is dramatic here is that although the $\mathbb{Q}(t)$-rank of (1.1) is 3, under

TABLE 3.1. The curves (1.1) with rank 7 occur at $t = 347, 443, 614, 757, 778, 784, 857, 877, 888$.

rank r	1	2	3	4	5	6	7
Number of curves (1.1) with rank r	1	2	220	413	272	82	9

specialization there are 22.02% of the curves with rank 3, 41.34% with rank 4, 27.23% with rank 5, 8.21% with rank 6, and 0.90% with rank 7. It was once believed that almost all curves under specialization would have the minimal rank consistent with the generic rank and sign of the functional equation, so here, ranks 3 or 4; yet ranks 3, 4 account for only 64% of the cases. Put another way there are 502 curves in the range with sign of functional equation -1; and 272 of these, or fully 54.18%, have rank 5, compared to 220 (43.82%) with the minimally consistent rank 3. Kramarz and Zagier [9] seem to give the first recorded instance of scepticism regarding minimality of the rank under specialization: in a family with generic rank 0, 23.3% of curves with sign of functional equation equal to +1 had rank at least 2 over the range searched. It must be stressed however that the number of curves considered here is really very small, and results of Brumer [1] interpret rather differently the voluminous data of Kramarz and Zagier. See also Brumer and McGuinness [2], and Rohrlich [6] for a more contemporary discussion of this issue. A further extraordinary feature of the family (1.1) is that in all but four cases computed, the Safarevic-Tate group has trivial 2-component, the exceptions being $t = 210, 285, 455, 645$, where the 2-component has order 4. I have no explanation for this phenomenon.

4. Several other similar pencils of curves may be written down with positive $\mathbb{Q}(t)$-rank, for example, the curves

$$E' : y^2 = x^3 - t^2x^2 + 1 \tag{4.1}$$

and its twist by i,

$$E'' : y^2 = x^3 + t^2x^2 - 1. \tag{4.2}$$

Similar computations as in Section 2 provide the following.

LEMMA 4.1. (i) *The curve E' at (4.1) has $\mathbb{Q}(t)$-rank 1 with generator $P = (0,1)$. The only points $(x, \pm y)$ of E' with $x, y \in \mathbb{Q}[t]$ correspond to P, $2P$, $4P$, namely $(0,1)$, $(t^2,1)$, $(t^8/4 - t^2, t^{12}/8 - t^6 + 1)$.*

(ii) *The curve E'' at (4.2) has $\mathbb{Q}(t)$-rank 1 with generator $P = (1,t)$. The only points $(x, \pm y)$ of E'' with $x, y \in \mathbb{Q}[t]$ correspond to $P, 3P$, namely,* $(1,t)$, $(64t^6/81 + 16t^4/9 + 8t^2/3 + 1, 512t^9/729 + 64t^7/27 + 16t^5/3 + 16t^3/3 + 3t)$.

The curves (4.1) and (4.2) are far less amenable to rank calculations with Cremona's "mrank" than the curves (1.1): search for the 2-descent curves is over a much larger region. We ranged over $1 \le t \le 50$; the largest rank of (4.1) is 4 at $t = 23$, the largest rank of (4.2) is 4 at $t = 36, 41, 46$.

References

[1] A. Brumer, *The average rank of elliptic curves. I*, Invent. Math. **109** (1992), no. 3, 445–472. MR 93g:11057. Zbl 783.14019.

[2] A. Brumer and O. McGuinness, *The behavior of the Mordell-Weil group of elliptic curves*, Bull. Amer. Math. Soc. (N.S.) **23** (1990), no. 2, 375–382. MR 91b:11076. Zbl 741.14010.

[3] D. A. Cox and S. Zucker, *Intersection numbers of sections of elliptic surfaces*, Invent. Math. **53** (1979), no. 1, 1–44. MR 81i:14023. Zbl 444.14004.

[4] M. Hindry and J. H. Silverman, *The canonical height and integral points on elliptic curves*, Invent. Math. **93** (1988), no. 2, 419–450. MR 89k:11044. Zbl 657.14018.

[5] M. Kuwata and J. Top, *An elliptic surface related to sums of consecutive squares*, Exposition. Math. **12** (1994), no. 2, 181–192. MR 95e:11068. Zbl 807.11017.

[6] D. E. Rohrlich, *Variation of the root number in families of elliptic curves*, Compositio Math. **87** (1993), no. 2, 119–151. MR 94d:11045. Zbl 791.11026.

[7] T. Shioda, *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Paul. **39** (1990), no. 2, 211–240. MR 91m:14056. Zbl 725.14017.

[8] J. H. Silverman, *Computing heights on elliptic curves*, Math. Comp. **51** (1988), no. 183, 339–358. MR 89d:11049. Zbl 656.14016.

[9] D. Zagier and G. Kramarz, *Numerical investigations related to the L-series of certain elliptic curves*, J. Indian Math. Soc. (N.S.) **52** (1987), 51–69 (1988). MR 90d:11072. Zbl 688.14016.

Bremner: Department of Mathematics, Arizona State University, Tempe, AZ 85287-1804, USA

E-mail address: bremner@asu.edu