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ELEMENTS IN EXCHANGE RINGS WITH RELATED COMPARABILITY

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ABSTRACT. We show that if R is an exchange ring, then the following are equivalent: (1) R satisfies related comparability. (2) Given $a,b,d \in R$ with $aR + bR = dR$, there exists a related unit $w \in R$ such that $a + bt = dw$. (3) Given $a,b \in R$ with $aR = bR$, there exists a related unit $w \in R$ such that $a = bw$. Moreover, we investigate the dual problems for rings which are quasi-injective as right modules.

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Let R be an associative ring with identity. From [6], R is said to satisfy related comparability provided that for any idempotents $e, f \in R$ with $e = 1 + ab$ and $f = 1 + ba$ for some $a, b \in R$, there exists a $u \in B(R)$ such that $ueR \lesssim^{\oplus} ufR$ and $(1-u)fR \lesssim^{\oplus} (1-u)eR$. The class of rings satisfying related comparability is quite large. It includes regular rings satisfying general comparability [10], one-sided unit regular rings [8] and partially unit-regular rings, while there still exist rings satisfying related comparability, which belong to none of the above classes (cf., [7, Example 10]).

In [4, 5], we studied related comparability over regular rings. In [6, 7], we investigated related comparability over exchange rings. It is shown that every exchange ring satisfying related comparability is separative [1]. Also, we show that related comparability over exchange rings is a Morita invariant. R is said to be an exchange ring if for every right R-module A and any two decompositions $A = M \oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, then there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\bigoplus_{i\in I} A_i')$. Many authors have investigated exchange rings with some kind of comparability properties so as to study problems related partial cancellation properties of modules (see [1, 2, 6, 7, 12, 13]).

In this paper, we investigate related comparability over exchange rings by related units. Recall that $w \in R$ is said to be a related unit of R if there exists some $e \in B(R)$ such that $w = eu + (1-e)v$ for some $u, v \in R$, where eu is right invertible in eR and $(1-e)v$ is left invertible in $(1-e)R$. $w \in R$ is said to be a semi-related unit of R if $w \in R$ is a related unit modulo $J(R)$. By virtue of semi-related units, we also give some new element-wise properties of rings which are quasi-injective as right modules.

Throughout, all rings are associative with identities. $B(R)$ denotes the set of all central idempotents of R and $\mathrm{r}\cdot\mathrm{ann}(b)\,(\mathrm{l}\cdot\mathrm{ann}(b))$ denotes the right (left) annihilator of $b \in R$.

Lemma 1. *Let R be an exchange ring. Then R satisfies related comparability if and only if so does the opposite ring R^{op} of R.*

Proof. Since R is an exchange ring, by virtue of [11, Proposition], so is the opposite ring R^{op} of R. Assume that R satisfies related comparability. Given $a^{\mathrm{op}}, b^{\mathrm{op}} \in R^{\mathrm{op}}$ with $a^{\mathrm{op}}x^{\mathrm{op}} + b^{\mathrm{op}} = 1^{\mathrm{op}}$, then we have $xa + b = 1$ in R. In view of [6, Theorem 4], there exists a $y \in R$ such that $x + by$ is a related unit of R. Thus, we have some $e \in B(R)$ such that $(x+by)e$ is right invertible in eR and $(x+by)(1-e)$ is left invertible in $(1-e)R$. By [5, Lemma 4], we claim that there are $z_1, z_2 \in R$ such that $(a+z_1b)e$ is left invertible in eR and $(a+z_2b)(1-e)$ is right invertible in $(1-e)R$. Let $z = z_1e + z_2(1-e)$. Then $a + zb$ is a related unit of R. Consequently, $a^{\mathrm{op}} + b^{\mathrm{op}}z^{\mathrm{op}}$ is a related unit of R^{op}. By [6, Theorem 4], we conclude that R^{op} satisfies related comparability. The converse is clear from $R \cong (R^{\mathrm{op}})^{\mathrm{op}}$. □

Theorem 2. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R satisfies related comparability.*

(2) *Given $a, b, d \in R$ with $aR + bR = dR$, there exists a related unit $w \in R$ such that $a + bt = dw$.*

(3) *Given a, b with $aR = bR$, there exists a related unit $w \in R$ such that $a = bw$.*

(4) *Given $a, b, d \in R$ with $Ra + Rb = Rd$, there exists a related unit $w \in R$ such that $a + tb = wd$.*

(5) *Given a, b with $Ra = Rb$, there exists a related unit $w \in R$ such that $a = wb$.*

Proof. (2)⟹(1). Trivial from [6, Theorem 4].

(1)⟹(2). Given $a, b, d \in R$ with $aR + bR = dR$. Let $g : dR \to dR/bR$ be the canonical map, $f_1 : R \to aR$ given by $r \mapsto ar$ for any $r \in R$, $f_2 : R \to bR$ given by $r \mapsto br$ for any $r \in R$, $f_3 : R \to dR$ given by $r \mapsto dr$ for any $r \in R$. Since $aR + bR = dR$, we know that gf_1, gf_3 are epimorphisms. On the other hand, R is a projective R-module. So there is some $\alpha \in \mathrm{End}_R R$ such that $gf_1 = gf_3\alpha$. Since gf_1 is a epimorphism, we also have some $\psi \in \mathrm{End}_R R$ such that $gf_3\alpha\psi = gf_3$. From $\alpha\psi + (1-\alpha\psi) = 1$, there is a $y \in \mathrm{End}_R R$ such that $\alpha + (1-\alpha\psi)y = w$ is a related unit of $\mathrm{End}_R R$. Therefore, we see that $gf_1 = gf_3\alpha = gf_3(\alpha + (1-\alpha\psi)y) = gf_3w$, and then $g(f_1 - f_3w) = 0$. Thus, we have $\mathrm{Im}(f_1 - f_3w) \le \mathrm{Ker}\, g = bR$. By the projectivity of right R-module R, there exists some $\beta \in \mathrm{End}_R R$ such that $f_2\beta = f_1 - f_3w$. Therefore, we claim that $a + b\beta(1) = f_1(1) + f_2(1)\beta(1) = f_3(1)w(1) = dw(1)$. It is easy to verify that $w(1)$ is a related unit of R.

(1)⟹(3). Given $a, b \in R$ with $aR = bR$, there exist $s, t \in R$ such that $a = bs$ and $b = at$. Thus, $b = bst$. Since $st + (1-st) = 1$, by virtue of [6, Theorem 4], there exists some $z \in R$ such that $s + (1-st)z = w$ is a related unit of R. Hence $a = bs = b(s + (1-st)z) = bw$, as desired.

(3)⟹(1). Given any regular $a \in R$. Then there exists some $b \in R$ such that $a = aba$, so $aR = abR$. Thus $a = abw$ for some related unit $w \in R$. Since $ab + (1-ab) = 1$, we see that $a + (1-ab)w = (ab + (1-ab))w = w$. By [5, Lemma 4], there is some $z \in R$ such that $b + z(1-ab) = m$ is a related unit of R. Hence $a = aba = a(b + z(1-ab))a = ama$. According to [6, Theorem 2], we claim that R satisfies related comparability.

(1)⟺(4)⟺(5). By [11, Proposition], we see that the opposite ring R^{op} of R is exchange. Using Lemma 1, we see that R satisfies related comparability if and only if so does the opposite ring R^{op} of R. Applying (1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3). To R^{op}, we easily derive the result. □

Corollary 3. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R satisfies related comparability.*

(2) *Given $a,b \in R$ with $aR + \mathrm{r}\cdot\mathrm{ann}(b) = R$, there exists some $k \in \mathrm{r}\cdot\mathrm{ann}(b)$ such that $a+k$ is a related unit.*

(3) *Given $a,b \in R$ with $Ra + \mathrm{l}\cdot\mathrm{ann}(b) = R$, there exists some $k \in \mathrm{l}\cdot\mathrm{ann}(b)$ such that $a+k$ is a related unit.*

Proof. (1)$\Longrightarrow$(2). Given $a,b \in R$ with $aR + \mathrm{r}\cdot\mathrm{ann}(b) = R$, then there exist $x \in R$, $k \in \mathrm{r}\cdot\mathrm{ann}(b)$ such that $ax + k = 1$. Since R satisfies related comparability, by virtue of [6, Theorem 4], we can find a $y \in R$ such that $a + ky$ is a related unit of R. It is easy to check that $ky \in \mathrm{r}\cdot\mathrm{ann}(b)$, as required.

(2)$\Longrightarrow$(1). Given $a,b \in R$ with $aR = bR$, there exist $s,t \in R$ such that $a = bs$ and $b = at$. Obviously, $1 - st \in \mathrm{r}\cdot\mathrm{ann}(b)$. Since $st + (1-st) = 1$, we have $sR + \mathrm{r}\cdot\mathrm{ann}(b) = R$. Thus we can find some $k \in \mathrm{r}\cdot\mathrm{ann}(b)$ such that $s + k = w$ is a related unit of R, and then $a = bs = b(s+k) = bw$, as asserted.

(1)$\Longleftrightarrow$(3). Trivial by the symmetry of related comparability. □

Recall that n is in the stable range of R provided that $a_1R + \cdots + a_{n+1}R = R$ with $a_1,\dots,a_{n+1} \in R$ implies that $(a_1 + a_{n+1}b_1)R + \cdots + (a_n + a_{n+1}b_n)R = R$ for some $b_1,\dots,b_n \in R$. If no such n exists, we say the stable range of R is ∞. $x \in R$ is said to be related unit-regular if $x = xwx$ for some related unit $w \in R$. Now, we investigate related comparability by related unit-regularity as follows.

Proposition 4. *Let R be an exchange ring with the finite stable range. Then the following are equivalent:*

(1) *R satisfies related comparability.*

(2) *Given $a,b,d \in R$ with $aR + bR = dR$, there exist some related unit-regular $w_1,w_2 \in R$ such that $aw_1 + bw_2 = d$.*

(3) *Given $a,b,d \in R$ with $Ra + Rb = Rd$, there exist some related unit-regular $w_1,w_2 \in R$ such that $w_1a + w_2b = d$.*

Proof. (1)$\Longrightarrow$(2). Given $aR + bR = dR$ with $a,b,d \in R$. For right R-module R^2, the two sets $\{a,b\}$ and $\{0,d\}$ generate the same right R-submodule of R^2. Thus, we can find $A,B \in M_2(R)$ such that $(a,b) = (0,d)A$, $(0,d) = (a,b)B$. Assume that $A = (a_{ij})$, $B = (b_{ij})$, $I_2 - AB = (c_{ij}) \in M_2(R)$. Since $AB + (I_2 - AB) = I_2$, we have $(a_{21},a_{22})(b_{12},b_{22})^T + c_{22} = 1$. Since R is an exchange ring satisfying related comparability, its stable range can only be $1,2$ or ∞ by [7, Theorem 3]. So 2 is in the stable range of R. Thus, we have some $(y_1,y_2) \in R^2$ such that $(a_{21},a_{22}) + c_{22}(y_1,y_2) \in R^2$ is unimodular. Set $Y = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \end{pmatrix}$. Then, we claim that the second row of $A + (I_2 - AB)Y = U$ is unimodular. Clearly, $(0,d)U = (0,d)A = (a,b)$. Since $u_{21}R + u_{22}R = R$, we can find orthogonal idempotents $e_1 \in u_{21}R$, $e_2 \in u_{22}R$ such that $e_1 + e_2 = 1$. Assume that $e_1 = u_{21}x_1$, $e_2 = u_{22}x_2$. Let $w_1 = x_1e_1$, $w_2 = x_2e_2$. Then w_1 and w_2 are both regular in R. Moreover, we have $u_{21}w_1 + u_{22}w_2 = 1$. By the related comparability of R, we claim that both w_i are related unit-regular, as asserted.

(2)$\Longrightarrow$(1). Given any regular $x \in R$. Then $x = xyx$ for a $y \in R$. So we have $xR + (1 - xy)R = R$, and then $xw_1 + (1 - xy)w_2 = 1$ for some related unit-regular $w_1, w_2 \in R$. We easily check that $x + (1 - xy)w_2 s \in R$ is related unit for some $s \in R$. Hence $y + t(1 - xy) = w$, i.e., a related unit of R. Consequently, we show that $x = xyx = xwx$, as desired.

(1)$\Longleftrightarrow$(3). Clear from the symmetry of related comparability. □

Recall that a module M is quasi-injective if any homomorphism of a submodule of M into M extends to an endomorphism of M. Now, we investigate rings which are quasi-injective as right modules. These extend the corresponding results in [3].

Lemma 5. *Let R be quasi-injective as a right R-module. Given $a, b \in R$ with $aR + bR = R$, there exists some $t \in R$ such that $a + bt$ is a semi-related unit.*

Proof. Given $a, b \in R$ with $aR + bR = R$, then $\bar{a}(R/J(R)) + \bar{b}(R/J(R)) = R/J(R)$. Since R is quasi-injective as a right R-module, by virtue of [9, Theorem 1], $R/J(R)$ is a regular, right self-injective ring. Hence R is an exchange ring satisfying related comparability. According to Theorem 2, we can find a $y \in R$ such that $\bar{a} + \bar{b}\bar{y} = \bar{w}$ is a related unit of $R/J(R)$. Therefore $a + by = w + r$ for some $r \in J(R)$. Clearly, $w + r$ is a semi-related unit of R, as desired. □

Theorem 6. *Let R be quasi-injective as a right R-module. Then the following hold:*

(1) *Given $a, b \in R$ with $\mathrm{r}\cdot\mathrm{ann}(a) = \mathrm{r}\cdot\mathrm{ann}(b)$, there exists a semi-related unit $w \in R$ such that $a = wb$.*

(2) *Given $a, b \in R$ with $\mathrm{l}\cdot\mathrm{ann}(a) = \mathrm{l}\cdot\mathrm{ann}(b)$, there exists a semi-related unit $w \in R$ such that $a = bw$.*

Proof. (1) Given $a, b \in R$ with $\mathrm{r}\cdot\mathrm{ann}(a) = \mathrm{r}\cdot\mathrm{ann}(b)$. Since R is quasi-injective as a right R-module, by [3, Lemma 3.2], we have $Ra = Rb$. Assume that $a = sb$, $b = ta$ for some $s, t \in R$. Then $b = tsb$. Consequently, there exists some $y \in R$ such that $t + (1 - ts)y$ is a semi-related unit of R by Lemma 5. Using [5, Lemma 4], we have some $z \in R$ such that $s + z(1 - ts) = w$ is a semi-related unit of R. Therefore, we claim that $a = sb = (s + z(1 - ts))b = wb$, as desired.

(2) Given $a, b \in R$ with $\mathrm{l}\cdot\mathrm{ann}(a) = \mathrm{l}\cdot\mathrm{ann}(b)$. Similarly to the consideration above, we have $aR = bR$. Assume that $a = bs$, $b = at$ for some $s, t \in R$. Then $b = bst$. From $st + (1 - st) = 1$, we can find a $y \in R$ such that $s + (1 - st)y = w$ is a semi-related unit of R. Therefore $a = bs = b(s + (1 - st)y) = bw$, whence the result. □

Corollary 7. *Let R be quasi-injective as a left R-module. Then the following hold:*

(1) *Given $a, b \in R$ with $\mathrm{r}\cdot\mathrm{ann}(a) = \mathrm{r}\cdot\mathrm{ann}(b)$, there exists a semi-related unit $w \in R$ such that $a = wb$.*

(2) *Given $a, b \in R$ with $\mathrm{l}\cdot\mathrm{ann}(a) = \mathrm{l}\cdot\mathrm{ann}(b)$, there exists a semi-related unit $w \in R$ such that $a = bw$.*

Proof. Applying Theorem 6 to the opposite ring R^{op} of R, we complete the proof. □

Theorem 8. *Let R be a ring which is quasi-injective as a right R-module. Then the following hold:*

(1) *Given* $a,b \in R$ *with* $\mathrm{r}\cdot\mathrm{ann}(a)\cap\mathrm{r}\cdot\mathrm{ann}(b)=0$, *there exists* $t\in R$ *such that* $a+tb$ *is a semi-related unit.*

(2) *Given* $a,b \in R$ *with* $\mathrm{l}\cdot\mathrm{ann}(a)\cap\mathrm{l}\cdot\mathrm{ann}(b)=0$, *there exists* $t\in R$ *such that* $a+bt$ *is a semi-related unit.*

Proof. (1) Given $a,b\in R$ with $\mathrm{r}\cdot\mathrm{ann}(a)\cap\mathrm{r}\cdot\mathrm{ann}(b)=0$, by virtue of [3, Proposition 3.4], we know that $Ra+Rb=R$. Thus, $(R/J(R))\bar{a}+(R/J(R))\bar{b}=R/J(R)$. Since R is a quasi-injective ring, from [9, Theorem 1], $R/J(R)$ is a regular, right self-injective ring. Moreover, we see that $R/J(R)$ satisfies related comparability. In view of Theorem 2, there exists $t\in R$ such that $\bar{a}+\bar{t}\bar{b}=\bar{w}$ with w is a semi-related unit of R. Thus, there is some $k\in J(R)$ such that $a+tb=w+k$. Clearly, $w+k$ is also a semi-related unit. Thus, we claim that $a+tb$ is a semi-related unit of R.

(2) Given $a,b\in R$ with $\mathrm{l}\cdot\mathrm{ann}(a)\cap\mathrm{l}\cdot\mathrm{ann}(b)=0$, analogously to [3, Proposition 3.4], we claim that $aR+bR=R$. Thus $\bar{a}(R/J(R))+\bar{b}(R/J(R))=R/J(R)$. Similarly to the consideration above, we show that $R/J(R)$ satisfies related comparability. In view of Theorem 2, there exists $t\in R$ such that $a+bt=w+k$ with w is a semi-related unit and $k\in J(R)$. Since $w+k$ is also a semi-related unit, the result follows. □

Corollary 9. *Let* R *be a ring which is quasi-injective as a left* R*-module. Then the following hold:*

(1) *Given* $a,b \in R$ *with* $\mathrm{r}\cdot\mathrm{ann}(a)\cap\mathrm{r}\cdot\mathrm{ann}(b)=0$, *there exists* $t\in R$ *such that* $a+tb$ *is a semi-related unit.*

(2) *Given* $a,b \in R$ *with* $\mathrm{l}\cdot\mathrm{ann}(a)\cap\mathrm{l}\cdot\mathrm{ann}(b)=0$, *there exists* $t\in R$ *such that* $a+bt$ *is a semi-related unit.*

Proof. Applying Theorem 8 to the opposite ring R^{op} of R, we easily obtain the result. □

Since every regular, right (left) self-injective ring is a quasi-injective ring with trivial Jacobson. As an immediate consequence of Theorem 6, Corollary 7, Theorem 8, and Corollary 9, we now derive the following.

Corollary 10. *Let* R *be a regular, right (left) self-injective ring. Then the following hold:*

(1) *Given* $a,b\in R$ *with* $\mathrm{r}\cdot\mathrm{ann}(a)=\mathrm{r}\cdot\mathrm{ann}(b)$, *there exists a related unit* $w\in R$ *such that* $a=wb$.

(2) *Given* $a,b\in R$ *with* $\mathrm{l}\cdot\mathrm{ann}(a)=\mathrm{l}\cdot\mathrm{ann}(b)$, *there exists a related unit* $w\in R$ *such that* $a=bw$.

(3) *Given* $a,b \in R$ *with* $\mathrm{r}\cdot\mathrm{ann}(a)\cap\mathrm{r}\cdot\mathrm{ann}(b)=0$, *there exists* $t\in R$ *such that* $a+tb$ *is a related unit.*

(4) *Given* $a,b \in R$ *with* $\mathrm{l}\cdot\mathrm{ann}(a)\cap\mathrm{l}\cdot\mathrm{ann}(b)=0$, *there exists* $t\in R$ *such that* $a+bt$ *is a related unit.*

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