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NORM ATTAINING BILINEAR FORMS ON $L_1(\mu)$

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ABSTRACT. Given a finite measure μ, we show that the set of norm attaining bilinear forms is dense in the space of all continuous bilinear forms on $L_1(\mu)$ if and only if μ is purely atomic.

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1. Introduction. A classical result of Bishop and Phelps [3] asserts that the set of norm attaining linear functionals on a Banach space is dense in the dual space. Very recently some attention has been paid to the question if the Bishop-Phelps theorem still holds for multilinear forms. To pose the problem more precisely, given a real or complex Banach space X and a natural number N, let us denote by $\mathcal{L}^N(X)$ the space of all continuous N-linear forms on X and let us say that $\varphi \in \mathcal{L}^N(X)$ attains its norm if there are $x_1,\dots,x_N \in B_X$ (the closed unit ball of X) such that

$$|\varphi(x_1,\dots,x_N)| = \|\varphi\| := \sup\{|\varphi(y_1,\dots,y_N)| : y_1,\dots,y_N \in B_X\}. \tag{1.1}$$

We denote by $\mathcal{AL}^N(X)$ the set of norm attaining continuous N-linear forms on X. The question is whether $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$ or not. Unlike the linear case, the answer to this question is negative, an example of a Banach space X such that $\mathcal{AL}^2(X)$ is not dense in $\mathcal{L}^2(X)$ was recently exhibited by Acosta, Aguirre, and Payá [1]. In [4], Choi gave a more striking counterexample by showing that $\mathcal{AL}^2(L_1[0,1])$ is not dense in $\mathcal{L}^2(L_1[0,1])$. In the positive direction, Aron, Finet, and Werner [2] showed that $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$ whenever X satisfies either the Radon-Nikodým property or the so-called property (α). Choi and Kim [5] obtained the same result for a Banach space X with a monotone shrinking basis and the Dunford-Pettis property (e.g., c_0). Jiménez and Payá [7] gave, for each N, an example of a Banach space X such that $\mathcal{AL}^N(X)$ is dense in $\mathcal{L}^N(X)$ but $\mathcal{AL}^{N+1}(X)$ is not dense in $\mathcal{L}^{N+1}(X)$, actually X is the canonical predual of a suitable Lorentz sequence space.

In this paper, we discuss the denseness of norm attaining multilinear forms on the space $L_1(\mu)$, where μ is an arbitrary finite measure. We show that $\mathcal{AL}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ (for all N, or just for $N = 2$) if and only if μ is purely atomic. Half of this characterization follows from the main result in [2], since $L_1(\mu)$ satisfies the Radon-Nikodým property if μ is purely atomic. For the converse, we first extend Choi's example [4] to show that $\mathcal{AL}^2(L_1(\mu))$ is not dense in $\mathcal{L}^2(L_1(\mu))$ where μ is the

product measure on an arbitrary product of copies of the unit interval. Then the result follows from the isometric classification of L_1-spaces (cf. [8]) through an elementary lemma which deals with the denseness of $\mathcal{AL}^2(X)$ when $X = Y \oplus_1 Z$ is the l_1-sum of two Banach spaces.

2. Results. In what follows $(\Omega, \mathcal{A}, \mu)$ will be a finite measure space. Let us start by recalling that the Banach space $\mathcal{L}^2(L_1(\mu))$ of all continuous bilinear forms on $L_1(\mu)$ is isometrically isomorphic to $L_\infty(\mu \otimes \mu)$, where $\mu \otimes \mu$ denotes the product measure on $\Omega \times \Omega$. More precisely, the bilinear form φ which corresponds to a function $h \in L_\infty(\mu \otimes \mu)$ is given by

$$\varphi(f,g) = \int_{\Omega\times\Omega} h(u,v) f(u) g(v) d\mu(u) d\mu(v), \tag{2.1}$$

for every $f, g \in L_1(\mu)$ (see [6]). Choi [4], has shown that $\mathcal{AL}^2(L_1[0,1])$ is not dense in $\mathcal{L}^2(L_1[0,1])$. This result can be extended in the following way.

Lemma 2.1. *Let ν be an arbitrary nonzero finite measure and $\mu = \nu \otimes m$, where m denotes Lebesgue measure on $I = [0,1]$. Then $\mathcal{AL}^2(L_1(\mu))$ is not dense in $\mathcal{L}^2(L_1(\mu))$.*

Proof. Let U be the set where ν is defined, so that μ works on $\Omega = I \times U$ and $\mu \otimes \mu$ lives on the set $\Omega \times \Omega = I \times U \times I \times U$. We want a function $h \in L_\infty(\mu \otimes \mu)$ such that the corresponding bilinear form cannot be approximated by norm attaining bilinear forms. Actually h will be the characteristic function χ_T of a suitable measurable set $T \subseteq \Omega \times \Omega$ with positive measure. The same argument used by Choi [4, Theorem 3] shows that the bilinear form corresponding to χ_T belongs to the closure of $\mathcal{AL}^2(L_1(\mu))$ only if there are measurable sets $E, F \subseteq \Omega$ with $\mu(E) > 0$, $\mu(F) > 0$, such that $[\mu \otimes \mu]$ $(E \times F \setminus T) = 0$. Therefore, we are left with finding a measurable set $T \subseteq \Omega \times \Omega$ with $[\mu \otimes \mu](T) > 0$ such that $[\mu \otimes \mu]$ $(E \times F \setminus T) > 0$ for any pair E, F of measurable subsets of Ω with $\mu(E) > 0$, $\mu(F) > 0$.

By [4, Lemma 2] there exists a set $S \subseteq I \times I$ with the analogous property. More concretely, S is a measurable set in $I \times I$, with $[m \otimes m](S) > 0$, such that $[m \otimes m](A \times B \setminus S)$ for any pair A, B of measurable subsets of I with $m(A) > 0$ and $m(B) > 0$. To get our set T, we modify S in the obvious way, namely we define

$$T = \{(s,u,t,v) \in I \times U \times I \times U : (s,t) \in S\}. \tag{2.2}$$

Clearly, T is a measurable set in $\Omega \times \Omega$, with positive measure. Let $E, F \subseteq \Omega$ be measurable sets in Ω with $\mu(E) > 0$, $\mu(F) > 0$, write $H = E \times F \setminus T$ and assume that $[\mu \otimes \mu](H) = 0$ to get a contradiction. For $u, v \in U$, let us consider the section

$$H^{(u,v)} = \{(s,t) \in I \times I : (s,u,t,v) \in H\}, \tag{2.3}$$

and note that $H^{(u,v)} = E^u \times F^v \setminus S$, where

$$E^u = \{s \in I : (s,u) \in E\}, \qquad F^v = \{t \in I : (t,v) \in F\}. \tag{2.4}$$

By Fubini's theorem (or the definition of the product measure) we have

$$0 = [m\otimes\nu\otimes m\otimes\nu](H) = \int_{U\times U}[m\otimes m](H^{(u,v)})d\nu(u)d\nu(v), \tag{2.5}$$

so

$$0 = [m\otimes m](H^{(u,v)}) = [m\otimes m](E^u\times F^v\setminus S), \tag{2.6}$$

for $[\nu\otimes\nu]$—almost every $(u,v)\in U\times U$. The property satisfied by S then implies that

$$m(E^u)m(F^v) = [m\otimes m](E^u\times F^v) = 0, \tag{2.7}$$

for $[\nu\otimes\nu]$—almost every $(u,v)\in U\times U$ and by applying to $E\times F$ the same argument used with H, we get

$$0 = [\mu\otimes\mu](E\times F) = \mu(E)\mu(F), \tag{2.8}$$

which is the required contradiction. □

Let us point out the special case of the above lemma that will be needed in the proof of our main result. Given an arbitrary nonempty set Λ, consider the product $[0,1]^\Lambda$ of so many copies of $[0,1]$ as indicated by Λ, with product measure μ. More concretely, μ is the unique positive normalized regular Borel measure on $[0,1]^\Lambda$ (provided with the product topology) such that, given a family $\{A_\lambda : \lambda\in\Lambda\}$ of Borel sets in $[0,1]$,

$$\mu\left(\prod_{\lambda\in\Lambda}A_\lambda\right) = \prod_{\lambda\in\Lambda}m(A_\lambda) = \inf\left\{\prod_{\lambda\in J}m(A_\lambda) : J\subset\Lambda,\ J \text{ finite}\right\}, \tag{2.9}$$

if all but countably many A_λ's are equal to $[0,1]$ and $\mu(\prod_{\lambda\in\Lambda}A_\lambda) = 0$ otherwise. The space $L_1(\mu)$ is usually denoted by $L_1([0,1]^\Lambda)$ (cf. [8, page 120]). By fixing $\lambda_0\in\Lambda$ and denoting by ν the product measure on $[0,1]^{\Lambda\setminus\{\lambda_0\}}$ we have clearly $\mu = m\otimes\nu$ and Lemma 2.1 tells us that $\mathcal{AL}^2(L_1[0,1]^\Lambda)$ is not dense in $\mathcal{AL}^2(L_1[0,1]^\Lambda)$.

We need another elementary lemma. By $Y\oplus_1 Z$ we denote the ℓ_1-sum of two Banach spaces Y and Z, i.e., $\|y+z\| = \|y\|+\|z\|$ for arbitrary $y\in Y$, $z\in Z$.

Lemma 2.2. *Let Y, Z be Banach spaces and $X = Y\oplus_1 Z$. If $\mathcal{AL}^2(X)$ is dense in $\mathcal{L}^2(X)$, then $\mathcal{AL}^2(Y)$ is dense in $\mathcal{L}^2(Y)$.*

Proof. Let $\varphi\in\mathcal{AL}^2(Y)$ with $\|\varphi\| = 1$ and $0<\epsilon<(1/2)$ be given. Define $\bar\varphi(x_1,x_2) = \varphi(Px_1,Px_2)$, $\forall x_1,x_2\in X$, where $P : X\to Y$ is the natural projection, and note that $\bar\varphi\in\mathcal{L}^2(X)$, $\|\bar\varphi\| = 1$. Since $\mathcal{AL}^2(X)$ is dense in $\mathcal{L}^2(X)$, there exists $\bar\psi\in\mathcal{AL}^2(X)$ such that $\|\bar\psi-\bar\varphi\|<\epsilon$ and $\|\bar\psi\| = 1$. Let $x_1 = y_1+z_1$, $x_2 = y_2+z_2$ with $y_1,y_2\in Y$, $z_1,z_2\in Z$ be such that $\|y_1\|+\|z_1\| = \|x_1\|\le 1$, $\|y_2\|+\|z_2\| = \|x_2\|\le 1$ and $|\bar\psi(x_1,x_2)| = 1$. Then we have

$$\begin{aligned}|\bar\psi(x_1,x_2)-\bar\psi(y_1,y_2)| &\le |\bar\psi(x_1,x_2)-\bar\varphi(x_1,x_2)| + |\bar\varphi(x_1,x_2)-\bar\psi(y_1,y_2)|\\ &= |\bar\psi(x_1,x_2)-\bar\varphi(x_1,x_2)| + |\bar\varphi(y_1,y_2)-\bar\psi(y_1,y_2)|\\ &\le \|\bar\psi-\bar\varphi\|(\|x_1\|\|x_2\|+\|y_1\|\|y_2\|) < 2\epsilon < 1\end{aligned} \tag{2.10}$$

hence $\bar\psi(y_1,y_2)\neq 0$, so $y_1\neq 0, y_2\neq 0$. Moreover we have

$$\begin{aligned} 1 = |\bar{\psi}(x_1,x_2)| &\le |\bar{\psi}(y_1,y_2)| + |\bar{\psi}(y_1,z_2)| + \bar{\psi}(z_1,y_2)| + |\bar{\psi}(z_1,z_2)| \\ &\le \|y_1\|\|y_2\| + \|y_1\|\|z_2\| + \|z_1\|\|y_2\| + \|z_1\|\|z_2\| \\ &= (\|y_1\| + \|z_1\|)(\|y_2\| + \|z_2\|) = \|x_1\|\|x_2\| \le 1, \end{aligned} \tag{2.11}$$

and it follows that

$$\left| \bar{\psi}\left(\frac{y_1}{\|y_1\|}, \frac{y_2}{\|y_2\|} \right) \right| = 1. \tag{2.12}$$

Now let ψ be the restriction of $\bar{\psi}$ to $Y \times Y$. Clearly $\|\psi\| \le 1$ and the above equality shows that $\psi \in \mathcal{AL}^2(Y)$. Note finally that $\|\psi - \varphi\| \le \|\bar{\psi} - \bar{\varphi}\| < \epsilon$. □

We are now ready for the main result.

Theorem 2.3. *Given a finite measure μ, the following statements are equivalent*
(1) *μ is purely atomic.*
(2) *$\mathcal{AL}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ for any natural number N.*
(3) *$\mathcal{AL}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ for some $N \ge 2$.*
(4) *$\mathcal{AL}^2(L_1(\mu))$ is dense in $\mathcal{L}^2(L_1(\mu))$.*

Proof. (1)$\Longrightarrow$(2). If μ is purely atomic, then $L_1(\mu)$ has the Radon-Nikodým property, and (2) follows from [2, Theorem 1].
(2)$\Longrightarrow$(3). This is trivial.
(3)$\Longrightarrow$(4). This is follows from [7, Proposition 2.1].
(4)$\Longrightarrow$(1). We use the isometric classification of L_1-spaces (see [8, Theorem 14.9]). Arguing by contradiction, we assume that (4) holds and that μ is not purely atomic. By the above-mentioned theorem, we can write $L_1(\mu)$ in the form

$$L_1(\mu) \cong L_1([0,1]^\Lambda) \oplus_1 Z, \tag{2.13}$$

for some nonempty set Λ and some Banach space Z. By Lemma 2.2 we get that $\mathcal{AL}^2(L_1([0,1]^\Lambda))$ is dense in $\mathcal{L}^2(L_1([0,1]^\Lambda))$, which contradicts Lemma 2.1. □

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