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ON A SUBGROUP OF THE AFFINE WEYL GROUP $\tilde{C}_4$

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ABSTRACT. We study a subgroup of the affine Weyl group $\tilde{C}_4$ and show that this subgroup is a homomorphic image of the triangle group $\triangle(3,4,4)$.

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1. Introduction. In the algebraic structures of the Coxeter groups $\tilde{A}_{n-1}, B_n, D_n$, we observe the following. $\tilde{A}_{n-1}$ is the subgroup of the wreath product $Z2S_n$ such that $\tilde{A}_{n-1} \cong Z^{n-1} \rtimes S_n$, where Z^{n-1} is the subgroup of Z^n consisting of all elements of exponent sum zero [2]; D_n is a subgroup of $B_n \cong Z2S_n$ such that $D_n \cong Z_2^{n-1} \rtimes S_n$ and Z_2^{n-1} is the subgroup of Z_2^n containing all elements of exponent sum zero [4]. We have the following natural question about $\tilde{C}_n \cong D_\infty^{n-1} \rtimes S_{n-1}$. What is the subgroup K of $\tilde{C}_n$, where $K \cong H \rtimes S_{n-1}$ and H is the subgroup of D_∞^{n-1} consisting of all elements of exponent sum zero [3]. In this paper we answer the question for $n = 4$ and find that the subgroup $H \rtimes S_3$ is a factor group of the triangle group $\triangle(3,4,4)$.

We begin by giving a presentation for the direct product of three copies of the infinite dihedral group

$$\begin{aligned} D_\infty^3 = \langle a_1, a_2, a_3, b_1, b_2, b_3 \mid\ & a_i^2 = b_i^2 = e, 1 \le i \le 3; \\ & a_i a_j = a_j a_i, 1 \le i < j \le 3; \\ & b_i b_j = b_j b_i, 1 \le i < j \le 3; \\ & a_i b_j = b_j b_i \text{ if } i \ne j, 1 \le i, j \le 3 \rangle. \end{aligned} \tag{1.1}$$

A presentation for the symmetric group of degree 3 is

$$S_3 = \langle x_1, x_2 \mid x_1^2 = x_2^2 = (x_1 x_2)^3 = e \rangle. \tag{1.2}$$

In [3], it is shown that $\tilde{C}_4$ is the semi-direct product $\tilde{C}_4 \cong D_\infty^3 \rtimes S_3$ with the natural action

$$(a_1, a_2, a_3)^{x_1} = (a_2, a_1, a_3), (a_1, a_2, a_3)^{x_2} = (a_1, a_3, a_2), \tag{1.3}$$

$$(b_1, b_2, b_3)^{x_1} = (b_2, b_1, b_3), (b_1, b_2, b_3)^{x_2} = (b_1, b_3, b_2). \tag{1.4}$$

We consider the subgroup H of D_∞^3 containing all elements of exponent sum zero. H is a normal subgroup of D_∞ and $D_\infty / H \cong \langle a_1 \mid a_1^2 = e \rangle$. Using the Reidemeister-Schreier process we find the following presentation for H:

$$H = \langle y_1, y_2, y_3, y_4, y_5 \mid y_1^2 = y_2^2 = y_3^2 = y_5^2 = (y_1y_2)^2 = (y_2y_3)^2 = (y_3y_4)^2 = (y_4y_5)^2 = (y_5y_1)^2 = (y_2y_4)^2 = (y_3y_5)^2 = (y_1y_4)^2 = e\rangle, \tag{1.5}$$

where $y_1 = a_1b_3, y_2 = a_2a_1, y_3 = a_1a_3, y_4 = a_1b_1, y_5 = a_1b_2$. From the action of S_3 on D_∞^3 we easily compute the following action of S_3 on H:

$$(y_1, y_2, y_3, y_4, y_5)^{x_1} = (y_2y_1, y_2, y_2y_3, y_2y_5, y_2y_4), \tag{1.6}$$

$$(y_1, y_2, y_3, y_4, y_5)^{x_2} = (y_5, y_3, y_2, y_4, y_1). \tag{1.7}$$

2. The group $H \rtimes S_3$. We use the method of presentation of group extensions described in [1] to find a presentation for $H \rtimes S_3$ with the action computed in Section 1. A presentation for $H \rtimes S_3$ is

$$H \rtimes S_3 = \langle x_1, x_2, y_1, y_2, y_3, y_4, y_5 \mid RH, RS_3, H^{S_3}\rangle, \tag{2.1}$$

where RH are the relations of H, RS_3 are the relations of S_3, the relations H^{S_3} are the action of S_3 on H. Lengthy computations using Tietze transformations give the following presentation for $H \rtimes S_3$,

$$H \rtimes S_3 = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^4 = (ca)^4 = (bacac)^3 = e\rangle. \tag{2.2}$$

We observe that if $\triangle(3,4,4)$ is the hyperbolic triangle group generated by a, b, and c and N is the normal closure of $(bcac)^3$ in $\triangle(3,4,4)$, then $H \rtimes S_3$ is the factor group $(\triangle(3,4,4))/N$.

3. The triangle group $\triangle(3,4,4)$. The triangle group $\triangle(3,4,4)$ is given by the presentations

$$\triangle(3,4,4) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^4 = (ca)^4 = e\rangle. \tag{3.1}$$

It is one of the hyperbolic triangle groups. $\triangle(3,4,4)$ is SQ-universal [6]. We find the derived subgroup of $\triangle(3,4,4)$ and show that it is SQ-universal using a method different from that in [7]. We also compute the growth series (word growth in the sense of Milnor and Gromov) of $\triangle(3,4,4)$. Using the Reidemeister-Schreier process we find that $\triangle'(3,4,4)$ is

$$\triangle'(3,4,4) = \langle x, y, z \mid x^2 = y^4 = (xy)^3 = (yz^{-1})^2 = e\rangle. \tag{3.2}$$

We consider the map $\theta : \triangle(3,4,4) \to Z_2 = \langle v \mid v^2 = e\rangle$ defined by $\theta(x) = \theta(y) = \theta(z) = v$. It is easy to see that

$$\ker\theta = \langle a, b, c, d \mid (ab)^2 = c^3 = d^3 = (ab^{-1})^2 = (bd^{-1})^2 = e\rangle. \tag{3.3}$$

We define another map $\phi : \ker\theta \to Z_2 = \langle u \mid u^2 = e\rangle$ by $\phi(a) = \phi(b) = u$ and $\phi(c) = \phi(d) = e$. Then $\ker\phi$ has the presentation

$$\ker\phi = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_3^2 = x_4^3 = x_5^3 = x_6^3 = (x_1x_2)^2 = (x_1x_4)^3 = x_2x_6^{-1}x_3x_5^{-1} = x_3x_5^{-1}x_2x_6^{-1} = e\rangle. \tag{3.4}$$

Letting $x_1 = x_5 = x_6 = e$ and $x_2 = x_3$ in $\ker\phi$ we get $\langle x_2, x_4 | x_2^2 = x_4^3 = e\rangle = Z_2 * Z_3$. Since the free product $Z_2 * Z_3$ is SQU [7], therefore $\ker\theta$ is SQU. But $\ker\theta$ is of finite index in $\triangle(3,4,4)$. Hence $\triangle(3,4,4)$ is SQU [7]. The growth series of $\triangle(3,4,4)$ is computed using exercise 26 in Section 1 of Chapter 4 in Bourbaki [5] as

$$\gamma(t) = \frac{(1+t)(1+t+t^2)(1+t+t^2+t^3)}{1-t^2-2t^3-t^4+t^6}. \tag{3.5}$$

We observe that zeros of the denominator of $\gamma(t)$ are not in the unit circle which implies that $\triangle(3,4,4)$ does not have a nilpotent subgroup of finite index. This is also known since $\triangle(3,4,4)$ is SQU.

Remark 3.1. It is interesting to know what subgroup of $\tilde{C}_n$ we get for $n > 4$. We did not find that yet.

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