Internat. J. Math. & Math. Sci.
Vol. 23, No. 5 (2000) 361–365
S0161171200001988

ALMOST AUTOMORPHIC SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS IN BANACH SPACES

GASTON MANDATA N'GUEREKATA

(Received 12 November 1997 and in revised form 2 March 1998)

ABSTRACT. We discuss the conditions under which bounded solutions of the evolution equation $x'(t) = Ax(t) + f(t)$ in a Banach space are almost automorphic whenever $f(t)$ is almost automorphic and A generates a C_0-group of strongly continuous operators. We also give a result for asymptotically almost automorphic solutions for the more general case of $x'(t) = Ax(t) + f(t, x(t))$.

Keywords and phrases. Almost automorphic functions, mild solutions, generator of a C_0-group, linear operators.

2000 Mathematics Subject Classification. Primary 34G10.

1. Introduction. Let A generate a C_0-group of strongly continuous operators $T(t)$, $t \in \mathbb{R}$ on a Banach space X. Let $f \in L^\infty(\mathbb{R}; X)$. A basic unsolved problem is: what is the structure of bounded (on $\mathbb{R}$) mild solutions of $x'(t) = Ax(t) + f(t)$? Classically results go back to Ordinary Differential Equations (when dimension of X is finite), and one sought solutions $x(t)$ such that $x(t) - y(t) \to 0$ as $t \to \infty$, when either $y(t)$ is a constant or a periodic function of time. In the evolution context of $x' = Ax + f$, much has been written on asymptotically constant or periodic solutions. Several authors extended these ideas to almost periodic solutions (when f is almost periodic). Our main result (Theorem 1.6) is inspired by the interesting work of Goldstein [3]. We are actually concerned with the more general case of almost automorphic, and when bounded solutions are almost automorphic. We also give a new result (Theorem 1.7) concerning mild solutions of the equation $x'(t) = Ax(t) + f(t, x(t))$ which approach almost automorphic functions at infinity under specific conditions on the function $f(t, x)$. See also [6] for another comparable situation.

Let X be a Banach space equipped with the topology norm and $\mathbb{R} = (-\infty, \infty)$ the set of real numbers. Let us first recall some definitions.

DEFINITION 1.1 (Bochner). A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if and only if, from any sequence of real numbers $(s'_n)_{n=1}^\infty$, we can subtract a subsequence $(s_n)_{n=1}^\infty$ such that: $\lim_{n\to\infty} f(t + s_n) = g(t)$ exists for each real number t, and $\lim_{n\to\infty} g(t - s_n) = f(t)$ for each t.

DEFINITION 1.2 [4]. A continuous function $f : \mathbb{R}^+ \to X$ is said to be asymptotically almost automorphic if and only if there exists an almost automorphic function $g : \mathbb{R} \to X$ and a continuous function $h : \mathbb{R}^+ \to X$ with $\lim_{t\to\infty} \|h(t)\| = 0$ and such that $f(t) = g(t) + h(t)$ for each $t \in \mathbb{R}^+$.

Definition 1.3. A Banach space X is said to be perfect if and only if every bounded function $u : \mathbb{R} \to X$ with an almost automorphic derivative $u'(t)$ is necessarily almost automorphic.

Remark 1.4. Uniformly convex Banach spaces are nice examples of perfect Banach spaces (see [10, Theorem 1.4]).

We consider the evolution equation

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}. \tag{1.1}$$

Theorem 1.5. *Let X be a perfect Banach space. Let A be a bounded linear operator $X \to X$ and $f : \mathbb{R} \to X$ an almost automorphic function. Then any bounded strong solution of (1.1) is almost automorphic if we assume that there exists a finite-dimensional subspace X_1 of X such that*

(α) $Ax(0) \in X_1$,

(β) $(e^{tA} - I)f(s) \in X_1$ *for any* $s, t \in \mathbb{R}$,

(γ) $e^{tA}u \in X_1$ *for any* $t \in \mathbb{R}$ *and for any* $u \in X_1$.

Proof. Let P be the projection of X onto X_1; such P always exists (cf. [7]) and possesses the following properties:

(1) $X = X_1 \oplus \ker(P)$, where $\ker(P)$ is the kernel of the operator P,

(2) P is bounded on X.

If we put $Q = I - P$, then it is easy to verify that $Q^2 = Q$ on X and $Qu = 0$ for any $u \in X_1$. Now if $x(t)$ is a bounded solution of (1.1), then we can write it as

$$x(t) = x_1(t) + x_2(t) \tag{1.2}$$

with $x_1(t) = Px(t) \in X_1$ and $x_2(t) = Qx(t) \in \ker(P)$.

Since $x(t)$ is bounded on $\mathbb{R}$, it is clear that both $x_1(t)$ and $x_2(t)$ are also bounded on $\mathbb{R}$. On the other hand, we have

$$x'(t) = x_1'(t) + x_2'(t) = Ax_1(t) + Ax_2(t) + Pf(t) + Qf(t), \quad t \in \mathbb{R}. \tag{1.3}$$

But $x(t)$ has the well-known Lagrange representation:

$$\begin{aligned} x(t) &= e^{tA}x(0) + \int_0^t e^{(t-s)A} f(s)\,ds \\ &= e^{tA}x(0) + \int_0^t f(s)\,ds + \int_0^t \left(e^{(t-s)A} - I\right) f(s)\,ds. \end{aligned} \tag{1.4}$$

By assumption (β), we deduce that $\int_0^t (e^{(t-s)A} - I)f(s)ds$ is in X_1, so that if we apply Q to both sides of (1.4), we get

$$x_2(t) = Qe^{tA}x(0) + Q\int_0^t f(s)\,ds = Qe^{tA}x(0) + \int_0^t Qf(s)\,ds, \tag{1.5}$$

consequently

$$x_2'(t) = Qe^{tA}Ax(0) + Qf(t) = Qf(t) \tag{1.6}$$

using conditions (α) and (γ).

It is clear that $Qf(t)$ and thus $x_2'(t)$ is almost automorphic (see [9, page 586]). Since $x_2(t)$ is bounded, then it is almost automorphic for we are in a perfect Banach space.

Now if we apply P to both sides of (1.3), we get in the finite-dimensional space X_1 the differential equation

$$x_1'(t) = PAx_1(t) + PAx_2(t) + P^2f(t) + PQf(t), \quad t \in \mathbb{R}. \tag{1.7}$$

Since the function $g(t) \equiv P^2f(t) + PQf(t)$ is almost automorphic and PA is a bounded linear operator, we deduce that $x_1(t)$ is almost automorphic [9, Theorem 3]. Finally, $x(t)$ is almost automorphic as the sum of two almost automorphic functions. □

Theorem 1.5 can be generalized to the case of unbounded operator A as follows.

Theorem 1.6. *In a perfect Banach space X, let A generate a C_0-group of strongly continuous linear operators $T(t)$, $t \in \mathbb{R}$. Assume that there exists a finite-dimensional subspace X_1 of X such that:*
(α) $Ax(0) \in X_1$,
(β') $(T(t)-I)f(s) \in X_1$ *for any* $s,t \in \mathbb{R}$,
(γ) $T(t)u \in X_1$ *for any* $t \in \mathbb{R}$ *and any* $u \in X_1$.
Then every bounded solution of (1.1) is almost automorphic.

Proof. We just follow the proof of Theorem 1.5 with the appropriate modifications. Here solutions are written as $x(t) = T(t)x(0) + \int_0^t T(t-s)f(s)ds$. □

We return now to a general (not necessarily perfect) Banach space X. We state and prove the following theorem.

Theorem 1.7. *Let A be a (possibly unbounded) linear operator which is the generator of a C_0-group of strongly continuous linear operators $T(t)$, $t \in \mathbb{R}$ such that $T(t)x : \mathbb{R} \to X$ is almost automorphic for each $x \in X$. Consider the differential equation*

$$x'(t) = Ax(t) + f(t,x(t)), \tag{1.8}$$

where $f(t,x) : \mathbb{R}\times X \to X$ is strongly continuous with respect to jointly t and x and such that $\|f(t,x) - f(t,y)\| \le L\|x-y\|$ for any $t \in \mathbb{R}$, $x,y \in X$, and $\int_0^\infty \|f(t,0)\|dt < \infty$.

Then every mild solution $x(t)$ of (1.8) with $\int_0^\infty \|x(t)\|dt < \infty$ is asymptotically almost automorphic.

Proof. Let $x : \mathbb{R}^+ \to X$ be a mild solution of (1.8). Then we have

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(s,x(s))\,ds. \tag{1.9}$$

We claim that $\int_0^\infty T(-s)f(s,x(s))ds$ exists in X (in Bochner's sense). Indeed, since $T(t)$ is almost automorphic for each $x \in X$, then

$$\sup_{t\in\mathbb{R}} \|T(t)x\| < \infty \quad \text{for each } x \in X. \tag{1.10}$$

Consequently

$$\sup_{t\in\mathbb{R}} \|T(t)\| = M < \infty, \tag{1.11}$$

by the uniform boundedness principle. Let us write

$$\int_0^\infty T(-s)f(s,x(s))\,ds = \int_0^\infty T(-s)\big(f(s,x(s)) - f(s,0)\big)\,ds + \int_0^\infty T(-s)f(s,0)\,ds, \tag{1.12}$$

then we get the inequality

$$\left\|\int_0^\infty T(-s)f(s,x(s))\,ds\right\| \le M\left(L\int_0^\infty \|x(s)\|\,ds + \int_0^\infty \|f(s,0)\|\,ds\right) < \infty. \tag{1.13}$$

Now the continuous function $F:\mathbb{R}\to X$ defined by

$$F(t) = \int_0^\infty T(t-s)f(s,x(s))\,ds = T(t)\int_0^\infty T(-s)f(s,x(s))\,ds \tag{1.14}$$

is almost automorphic; therefore $V(t) = T(t)x(0)+F(t)$ is also almost automorphic. Let us consider the continuous function $W:\mathbb{R}^+\to X$

$$W(t) = -\int_t^\infty T(t-s)f(s,x(s))\,ds. \tag{1.15}$$

If we use the same computation as for $F(t)$ in (1.14), we get

$$\|W(t)\| \le M\left(L\int_t^\infty \|x(s)\,ds\| + \int_t^\infty \|f(s,0)\,ds\|\right) \tag{1.16}$$

which shows that $\lim_{t\to\infty}\|W(t)\| = 0$.

Finally $x(t) = V(t)+W(t)$, $t\in\mathbb{R}^+$ is asymptotically almost automorphic. □

Remark 1.8. (1) An example of Theorem 1.5 (occurring in Sturm-Liouville theory, for instance) is when X is a Hilbert space and $A\varphi_n = \lambda_n\varphi_n$ for $\{\varphi_n : n = 1,2,\dots\}$ an orthonormal basis and $|\mathrm{Re}(\lambda_n)| \le M$ for all n. For X_1, one may take $X_1 = \mathrm{span}\{\varphi_1,\dots,\varphi_N\}$ (for any N) and assume $f\in L^\infty(\mathbb{R},X_1)$.

(2) An example of operator A satisfying the hypothesis of Theorem 1.7 is the above example with $A^* = -A$, i.e., $M = 0$.

Acknowledgements. The present research was supported by a research grant from Morgan State University. The author is very grateful to the referee for his/her useful suggestions.

References

[1] S. Bochner, *Uniform convergence of monotone sequences of functions*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 582–585. MR 23#A3390. Zbl 103.05304.

[2] ———, *Continuous mappings of almost automorphic and almost periodic functions*, Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 907–910. MR 29#6252. Zbl 134.30102.

[3] J. A. Goldstein, *Convexity, boundedness, and almost periodicity for differential equations in Hilbert space*, Internat. J. Math. Math. Sci. **2** (1979), no. 1, 1–13. MR 80e:34040. Zbl 397.34041.

[4] G. M. N'Guérékata, *Sur les solutions presqu'automorphes d'équations différentielles abstraites*, Ann. Sci. Math. Québec **5** (1981), no. 1, 69–79. MR 82h:34085. Zbl 494.34045.

[5] ———, *Quelques remarques sur les fonctions asymptotiquement presque automorphes*, Ann. Sci. Math. Québec **7** (1983), no. 2, 185–191. MR 84k:43009. Zbl 524.34064.

[6] ———, *An asymptotic theorem for abstract differential equations*, Bull. Austral. Math. Soc. **33** (1986), no. 1, 139–144. MR 87c:34126. Zbl 581.34029.

[7] M. Schechter, *Principles of Functional Analysis*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1973. MR 57#7085.

[8] W. A. Veech, *Almost automorphic functions on groups*, Amer. J. Math. **87** (1965), 719–751. MR 32#4469. Zbl 137.05803.

[9] S. Zaidman, *Almost automorphic solutions of some abstract evolution equations*, Istit. Lombardo Accad. Sci. Lett. Rend. A **110** (1976), no. 2, 578–588 (1977). MR 58#6593. Zbl 374.34042.

[10] M. Zaki, *Almost automorphic solutions of certain abstract differential equations*, Ann. Mat. Pura Appl. (4) **101** (1974), 91–114. MR 51#1059. Zbl 304.42028.

N'Guerekata: Department of Mathematics, Morgan State University, Cold Spring Lane and Hillen Road, Baltimore, MD 21251, USA


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization