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# COMMON STATIONARY POINTS OF MULTIVALUED MAPPINGS ON BOUNDED METRIC SPACES

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Abstract. Necessary and sufficient conditions for the existence of common stationary points of two multivalued mappings and common stationary point theorems for multivalued mappings on bounded metric spaces are given. Our results extend the theorems due to Fisher in 1979, 1980, and 1983 and Ohta and Nikaido in 1994.

Keywords and phrases. Common stationary point, multivalued mappings, complete bounded metric space.

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**1. Introduction.** Let $(X,d)$ be a metric space and $B(X)$ denote the set of all nonempty bounded subsets of $X$. For $A$, $B \in X$, let $\delta(A,B) = \sup\{d(a,b) : a \in A,\ b \in B\}$ and $\delta(A) = \delta(A,A)$. If $A$ consists of a single point $a$, we write $\delta(A,B) = \delta(a,B)$. If $B$ also consists of a single point $b$, we write $\delta(A,B) = \delta(a,b) = d(a,b)$. Let $N$ and $\omega$ denote the sets of positive integers and nonnegative integers, respectively. Let $\Phi$ denote a family of mappings such that for each $\phi \in \Phi$, $\phi : [0,\infty) \to [0,\infty)$ is upper semicontinuous, nondecreasing and $\phi(t) < t$ for $t > 0$.

The following definitions and lemmas were introduced by Fisher [3] and Singh and Meade [6].

**Definition 1.1** [3]. Let $\{A_n\}$ be a sequence of sets in $B(X)$ and $A \in B(X)$. The sequence $\{A_n\}$ is said to *converge* to the set $A$ if
- (i) each point $a \in A$ is the limit of some convergent sequence $\{a_n\}$, where $a_n \in A_n$ for $n \in N$;
- (ii) for arbitrary $\epsilon > 0$, there exists $k \in N$ such that $A_n \subseteq A_\epsilon$, for $n > k$, where $A_\epsilon$ is the union of all open spheres with centres in $A$ and radius $\epsilon$.

**Definition 1.2** [3]. Let $F$ be a multivalued mapping of $(X,d)$ into $B(X)$. The mapping $F$ is called *continuous* in $X$ if whenever $\{x_n\}$ is a sequence of points in $X$ converging to $x \in X$, the sequence $\{Fx_n\}$ in $B(X)$ converges to $Fx \in B(X)$.

**Lemma 1.3** [3]. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space $(X,d)$ which converge to the bounded subsets $A$ and $B$, respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

**Lemma 1.4** [6]. *Let $\phi \in \Phi$. Then $\phi(t) < t$ for each $t > 0$ if and only if*

$$\lim_{n\to\infty} \phi^n(t) = 0, \tag{1.1}$$

*where $\phi^n$ denotes the $n$-times composition of $\phi$.*

Let $F$ and $G$ be mappings of $(X,d)$ into $B(X)$. A point $x \in X$ is called a *common stationary point* of $F$ and $G$ if $Fx = Gx = \{x\}$. For $A \subseteq X$, let $FA = \cup_{a\in A} Fa$ and $GFA = G(FA)$. The mappings $F$ and $G$ are said to *commute* if $FGx = GFx$ for $x \in X$. Define $C_F = \{T : T \text{ is a mapping of } X \text{ into } B(X) \text{ and } T \text{ and } F \text{ commute}\}$ and $CC_F = \{T : T \text{ is continuous and } T \in C_F\}$. It follows that $C_F \supseteq \{F^n : n \in \omega\}$, where $F^0x = \{x\}$ for $x \in X$.

Throughout the rest of the paper, we assume that $(X,d)$ is a complete bounded metric space.

In 1979, Fisher [1] proved a common fixed point theorem for commuting mappings $f$ and $g$ of $(X,d)$ into itself satisfying

$$d(fx,gy) \le c\max\{d(x,y),d(x,fx),d(y,gy),d(x,gy),d(y,fx)\} \tag{1.2}$$

for all $x$, $y \in X$, where $0 \le c < 1$.

In 1980, Fisher [2] generalized the result to multivalued mappings $F$ and $G$ of $(X,d)$ into $B(X)$ satisfying the condition

$$\delta(Fx,Gy) \le c\max\{\delta(x,y),\delta(x,Fx),\delta(y,Gy),\delta(x,Gy),\delta(y,Fx)\} \tag{1.3}$$

for all $x$, $y \in X$, where $0 \le c < 1$.

In 1983, Fisher [4] established a common fixed point theorem for continuous, commuting mappings $F$ and $G$ of $(X,d)$ into $B(X)$ satisfying

$$\begin{aligned}\delta(F^pG^px,F^pG^py) \le c\max\{&\delta(F^qG^rx,F^sG^ty),\delta(F^qG^rx,F^sG^tx),\\ &\delta(F^qG^ry,F^sG^ty) : 0 \le q,r,s,t \le p\}\end{aligned} \tag{1.4}$$

for all $x$, $y \in X$, where $0 \le c < 1$ and $p$ is a fixed positive integer.

In 1994, Ohta and Nikaido [5] obtained the existence of fixed point for a continuous self mapping $f$ of $(X,d)$ satisfying

$$d(f^kx,f^ky) \le c\delta(\{f^it : t \in \{x,y\},\ i \in \omega\}) \tag{1.5}$$

for all $x$, $y \in X$, where $0 \le c < 1$ and $k$ is a fixed positive integer.

The first purpose of the paper is to establish criteria for the existence of common stationary points of commuting mappings $F$ and $G$ of $(X,d)$ into $B(X)$. The second purpose of the paper is to prove common stationary point theorems for commuting mappings $F$ and $G$ of $(X,d)$ into $B(X)$ satisfying one of the following:

$$\delta(F^pG^px,F^qG^qy) \le \phi(\delta(\cup_{D\in C_{FG}} D\{x,y\})) \tag{1.6}$$

for all $x$, $y \in X$, where $\phi \in \Phi$ and $p,q$ are fixed positive integers;

$$\delta(F^px,G^qy) \le \phi(\delta(\cup_{D\in C_F\cap C_G} D\{x,y\})) \tag{1.7}$$

for all $x$, $y \in X$, where $\phi \in \Phi$ and $p,q$ are fixed positive integers;

$$\delta(Fx,Gy) \le \phi(\max\{\delta(x,Fx),\delta(y,Gy),\delta(x,Gy),\delta(y,Fx),\delta(\cup_{D\in CC_{FG}} D\{x,y\})\}) \tag{1.8}$$

for all $x$, $y \in X$, where $\phi \in \Phi$.

It is easy to see that (1.2) and (1.3) are special cases of (1.8), that (1.4) and (1.5) are special cases of (1.6), and that (1.2) and (1.5) are special cases of (1.7). Our results extend and unify the theorems of Fisher [1, 2, 4] and Ohta and Nikaido [5].

## 2. Common stationary points.

Our main results are as follows.

**Theorem 2.1.** *Let $F$ and $G$ be continuous and commuting mappings of $(X,d)$ into $B(X)$. Then the following statements are equivalent:*

(i) *$F$ and $G$ have a common stationary point;*

(ii) *there exist $S,T \in CC_F \cap CC_G$ with $S \in C_T$ and $\phi \in \Phi$ such that*

$$\delta(Sx,Ty) \le \phi(\delta(\cup_{D\in CC_S\cap CC_T} D\{x,y\})) \quad \forall x,y \in X; \tag{2.1}$$

(iii) *there exist $S,T \in C_F \cap C_G$ with $S \in C_T$ and $\phi \in \Phi$ such that*

$$\delta(Sx,Ty) \le \phi(\delta(\cup_{D\in C_S\cap C_T} D\{x,y\})) \quad \forall x,y \in X; \tag{2.2}$$

(iv) *there exist mappings $S,T$ of $(X,d)$ into $B(X)$ with $S \in C_T$ and $\phi \in \Phi$ such that $F,G \in C_{ST}$ and*

$$\delta(Sx,Ty) \le \phi(\delta(\cup_{D\in C_{ST}} D\{x,y\})) \quad \forall x,y \in X. \tag{2.3}$$

**Proof.** We shall verify the following implications: (i)⇒(ii)⇒(iii)⇒(iv)⇒(i). Suppose, first of all, that $F$ and $G$ have a common stationary point $z$. Define mappings $S$ and $T$ of $(X,d)$ into $B(X)$ by $Sx = Tx = \{z\}$ for all $x \in X$. It is easy to check that $S,T \in CC_F \cap CC_G$ and

$$\delta(Sx,Ty) = 0 \le \phi(\delta(\cup_{D\in CC_S\cap CC_T} D\{x,y\})) \tag{2.4}$$

for all $x$, $y \in X$, $\phi \in \Phi$, that is, (ii) holds.

Note that $CC_F \subseteq C_F$ and $C_S \cap C_T \subseteq C_{ST}$. Therefore (ii)⇒(iii)⇒(iv) are clear.

We now assume that (iv) holds. Then for any $A$, $B \in B(X)$, we have

$$\delta(SA,TB) \le \phi(\delta(\cup_{D\in C_{ST}} D(A\cup B))), \tag{2.5}$$

by (iv). Since $X$ is bounded, $M = \delta(X) < \infty$. Set $X_n = S^nT^nX$ for $n \in N$. Then $X_n \subseteq X_{n-1}$ for $n \in N$. We now will prove by induction that

$$\delta(X_n) \le \phi^n(M) \quad \text{for } n \in N. \tag{2.6}$$

Note that $S$ and $T$ commute. From (2.5), we have

$$\delta(X_1) = \delta(STX,TSX) \le \phi(\delta(\cup_{D\in C_{ST}} D(TX\cup SX))) \le \phi(\delta(X)) = \phi(M), \tag{2.7}$$

that is, (2.6) holds for $n = 1$. Assume now that (2.6) holds for some $n \in N$. It follows from (2.5) that

$$\begin{aligned} \delta(X_{n+1}) &= \delta(S^{n+1}T^{n+1}X,T^{n+1}S^{n+1}X) \\ &\le \phi(\delta(\cup_{D\in C_{ST}} D(S^nT^{n+1}X\cup T^nS^{n+1}X))) \\ &= \phi(\delta(\cup_{D\in C_{ST}} S^nT^n(DTX\cup DSX))) \\ &\le \phi(\delta(X_n)) \le \phi^{n+1}(M) \end{aligned} \tag{2.8}$$

by our assumption. Hence (2.6) follows by induction. Choose $x_n \in X_n$ for $n \in N$. Then, by (2.6), we get

$$d(x_n,x_m) \le \delta(X_n,X_m) \le \delta(X_n) \le \phi^n(M) \quad \text{for } m > n. \tag{2.9}$$

Consequently, $\{x_n\}$ is a Cauchy sequence by Lemma 1.4. Since $X$ is complete, there exists a point $z$ in $X$ such that $x_n \to z$ as $n \to \infty$. From (2.6), we have

$$\delta(z,X_n) \le d(z,x_m) + \delta(x_m,X_n) \le d(z,x_m) + \delta(X_n) \le d(z,x_m) + \phi^n(M) \tag{2.10}$$

for $m,n \in N$ with $m > n$. Letting $m$ tend to infinity, we obtain

$$\delta(z,X_n) \le \phi^n(M) \quad \text{for } n \in N. \tag{2.11}$$

Since $F$ is continuous and $x_n \to z$, then $\{Fx_n\}$ converges to $\{Fz\}$. Note that

$$Fx_n \subseteq FS^nT^nX = S^nT^nFX \subseteq X_n \quad \text{for } n \in N. \tag{2.12}$$

Then $\delta(z,Fx_n) \le \delta(z,X_n)$ for $n \in N$. Letting $n$ tend to infinity, we have $\delta(z,Fz) \le 0$ by (2.11) and Lemmas 1.3 and 1.4, that is, $Fz = \{z\}$. Similarly, we have $Gz = \{z\}$. This completes the proof. □

**Theorem 2.2.** *Let $F$ and $G$ be continuous and commuting mappings of $(X,d)$ into $B(X)$ satisfying (1.6) or (1.7). Then $F$ and $G$ have a unique common stationary point $z$ and the sequence $\{F^nG^nx\}$ converges to $\{z\}$ for all $x \in X$.*

**Proof.** Let $M = \delta(X)$, $k = p+q$, $X_n = F^nG^nX$ and $x_n \in X_n$ for $n \in N$. Note that every $n \in N$ can be written as

$$n = kj + i, \tag{2.13}$$

where $j \in \omega$ and $0 \le i < k$. Now we claim that

$$\delta(X_n) \le \phi^j(M). \tag{2.14}$$

If (1.6) is satisfied, then

$$\begin{aligned}
\delta(X_n) &= \delta(F^pG^p(F^{q+i}G^{q+i}X_{k(j-1)}),F^qG^q(F^{p+i}G^{p+i}X_{k(j-1)}))\\
&\le \phi(\delta(\cup_{D\in C_{FG}} D(F^{q+i}G^{q+i}X_{k(j-1)} \cup F^{p+i}G^{p+i}X_{k(j-1)})))\\
&= \phi(\delta(\cup_{D\in C_{FG}} (F^{k(j-1)}G^{k(j-1)}F^{q+i}G^{q+i}DX \cup F^{k(j-1)}G^{k(j-1)}F^{p+i}G^{p+i}DX)))\\
&\le \phi(\delta(X_{k(j-1)}))
\end{aligned} \tag{2.15}$$

which implies that

$$\delta(X_{kj}) \le \phi(\delta(X_{k(j-1)})) \le \cdots \le \phi^{j-1}(\delta(X_k)) \le \phi^j(M). \tag{2.16}$$

Note that $X_n \subseteq X_{n-1}$. Thus (2.14) follows from (2.15) and (2.16). If (1.7) is satisfied, then

$$\begin{aligned}
\delta(X_n) &= \delta(F^p(F^{q+i}G^{k+i}X_{k(j-1)}),G^q(F^{k+i}G^{p+i}X_{k(j-1)}))\\
&\le \phi(\delta(\cup_{D\in C_F\cap C_G} D(F^{q+i}G^{k+i}X_{k(j-1)} \cup F^{k+i}G^{p+i}X_{k(j-1)})))\\
&\le \phi(\delta(X_{k(j-1)})).
\end{aligned} \tag{2.17}$$

Similarly, (2.16) holds also. It follows from (2.16) and (2.17) that (2.14) holds.

Given $x_n \in X_n$ for all $n \in N$. For any $m > n > k$, by (2.13) and (2.14) we have

$$d(x_n, x_m) \le \delta(X_n) \le \phi^j(M). \tag{2.18}$$

As in the proof of Theorem 2.1, we conclude that $F$ and $G$ have a common stationary point $z$ and that $x_n \to z$ as $n \to \infty$. Suppose that $F$ and $G$ have a second common stationary point $w$. Then $\{u\} = F^n G^n u \subseteq X_n$ for $u \in \{z, w\}$ and $n \in N$. In view of (2.13) and (2.14), we infer that

$$d(z, w) \le \delta(X_n) \le \phi^j(M). \tag{2.19}$$

Letting $n$ tend to infinity we have $d(z, w) \le 0$ by Lemma 1.4, that is, $z = w$. Hence $F$ and $G$ have a unique common stationary point $z$. For $x \in X$ and $n \in N$, choose $y_n \in F^n G^n x$. Using (2.13) and (2.14), we have

$$d(y_n, z) \le \delta(F^n G^n x, z) \le \delta(X_n, z) \le \delta(X_n) \le \phi^j(M). \tag{2.20}$$

Letting $n$ tend to infinity, by Lemma 1.4 and Definition 1.1 and the above inequalities, we conclude that $\{F^n G^n x\}$ converges to $\{z\}$. This completes the proof. □

As a consequence of Theorem 2.2, we have the following corollary.

**Corollary 2.3.** *Let $F$ and $G$ be continuous and commuting mappings of $(X, d)$ into $B(X)$ satisfying one of the following:*

$$\delta(F^q x, G^q y) \le \phi(\delta(\cup_{i,j\in\omega} F^i G^j \{x, y\})) \quad \forall x, y \in X, \tag{2.21}$$

*where $\phi \in \Phi$ and $p, q$ are fixed positive integers;*

$$\delta(F^p G^p x, F^q G^q y) \le \phi(\delta(\cup_{i,j\in\omega} F^i G^j \{x, y\})) \quad \forall x, y \in X, \tag{2.22}$$

*where $\phi \in \Phi$ and $p, q$ are fixed positive integers. Then $F$ and $G$ have a unique common stationary point $z$ and the sequence $\{F^n G^n x\}$ converges to $\{z\}$ for all $x \in X$.*

From Corollary 2.3, we have the following.

**Corollary 2.4** [4, Theorem 1]. *Let $F$ and $G$ be continuous and commuting mappings of $(X, d)$ into $B(X)$ satisfying (1.3). Then $F$ and $G$ have a unique common stationary point $z$ and the sequence $\{F^n G^n x\}$ converges to $\{z\}$ for all $x \in X$.*

**Corollary 2.5** [5, Theorem 3]. *Let $f$ be a continuous mapping of $(X, d)$ into itself satisfying (1.5). Then $f$ has a unique fixed point $z$ and for each $x \in X$, $f^n x \to z$ as $n \to \infty$.*

**Theorem 2.6.** *Let $F$ and $G$ be commuting mappings of $(X, d)$ into $B(X)$ satisfying (1.8). Then $F$ and $G$ have a unique common stationary point $z$ and the sequence $\{F^n G^n x\}$ converges to $\{z\}$ for all $x \in X$. Further, $\{z\} = Dz$ for all $D \in CC_{FG}$.*

**Proof.** Let $M = \delta(X)$, $X_n = F^n G^n X$, and $x_n \in X_n$ for $n \in N$. As in the proof of Theorem 2.1, we conclude that $\delta(X_n) \le \phi^n(M)$ for $n \in N$ and that $x_n \to z$, $\delta(z, X_n) \to$

0 as $n \to \infty$. Consequently, the sequences $\{X_n\}$ and $\{\{z\} \cup X_n\}$ converge to $\{z\}$. For each $D \in CC_{FG}$, we have $\delta(Dx_n, z) \to \delta(Dz, z)$ as $n \to \infty$, by the continuity of $D$ and Lemma 1.3. Note that

$$\begin{aligned} \delta(Dx_n, z) &\le \delta(Dx_n, x_n) + d(x_n, z) \le \delta(X_n) + d(x_n, z) \\ &\le \phi^n(M) + d(x_n, z) \longrightarrow 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2.23}$$

which implies that $\delta(Dz, z) \le 0$, that is, $Dz = \{z\}$. Using (1.8), we have for $n \in N$,

$$\begin{aligned} &\delta(F^n G^n X, Gz) \\ &\quad \le \phi\big(\max\{\delta(F^{n-1}G^n X, F^n G^n X), \delta(z, Gz), \delta(F^{n-1}G^n X, Gz), \\ &\qquad\qquad \delta(z, F^n G^n X), \delta(\cup_{D \in CC_{FG}} D(F^{n-1}G^n X \cup \{z\}))\}\big) \\ &\quad \le \phi\big(\max\{\delta(X_{n-1}, X_n), \delta(z, Gz), \delta(X_{n-1}, Gz), \\ &\qquad\qquad \delta(z, X_n), \delta(\cup_{D \in CC_{FG}} (F^{n-1}G^{n-1}DGX \cup Dz))\}\big) \\ &\quad \le \phi\big(\max\{\delta(X_{n-1}), \delta(z, Gz), \delta(X_{n-1}, Gz), \\ &\qquad\qquad \delta(z, X_n), \delta(X_{n-1} \cup \{z\})\}\big), \end{aligned} \tag{2.24}$$

which implies that

$$\begin{aligned} \delta(x_n, Gz) &\le \delta(F^n G^n X, Gz) \\ &\le \phi\big(\max\{\delta(z, Gz), \delta(X_{n-1}, Gz), \delta(z, X_n), \delta(X_{n-1} \cup \{z\})\}\big). \end{aligned} \tag{2.25}$$

Letting $n$ tend to infinity, we get

$$\delta(z, Gz) \le \phi\big(\max\{\delta(z, Gz), \delta(z, Gz), 0, 0\}\big) = \phi(\delta(z, Gz)). \tag{2.26}$$

Suppose that $\delta(z, Gz) > 0$. Then

$$\delta(z, Gz) \le \phi(\delta(z, Gz)) < \delta(z, Gz), \tag{2.27}$$

which is a contradiction. Therefore $\delta(z, Gz) = 0$, that is, $Gz = \{z\}$. Similarly we have $Fz = \{z\}$. The rest of the proof is exactly the same as that of Theorem 2.2. This completes the proof. □

From Theorem 2.6, we have the following corollary.

**Corollary 2.7** [2, Theorem 2]. *Let $F$ and $G$ be commuting mappings of $(X, d)$ into $B(X)$ satisfying (1.3). Then $F$ and $G$ have a unique common stationary point $z$ and the sequence $\{F^n G^n x\}$ converges to $\{z\}$ for all $x \in X$.*

**Corollary 2.8** [1, Theorem 4]. *Let $f$ and $g$ be commuting mappings of $(X, d)$ into itself satisfying (1.2). Then $f$ and $g$ have a unique common fixed point $z$ and for each $x \in X$, $f^n g^n \to z$ as $n \to \infty$.*

The following example shows that Theorem 2.6 extends properly Corollaries 2.7 and 2.8.

**Example 2.9.** Let $X = \{1,2,5,8\}$ with the usual metric. Define self mappings $f$ and $g$ of $(X,d)$ by

$$f1 = 1, \qquad f2 = f5 = g1 = g2 = g5 = 5, \qquad f8 = g8 = 2. \tag{2.28}$$

Set $Fx = \{fx\}$ and $Gx = \{gx\}$ for $x \in X$. Let $\phi(t) = (1/2)t$ for $t \geq 0$. It is easy to check that $F$ and $G$ satisfy the conditions of Theorem 2.6. But Corollaries 2.7 and 2.8 are not applicable since

$$d(f1,g1) = 4 = \max\{d(1,1),d(1,f1),d(1,g1),d(1,g1),d(1,f1)\}, \tag{2.29}$$

that is, $f$ and $g$ do not satisfy (1.2). Similarly $F$ and $G$ do not satisfy (1.3).

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