RELATED FIXED POINTS FOR SET VALUED MAPPINGS ON TWO METRIC SPACES

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ABSTRACT. Some related fixed points theorems for set valued mappings on two complete and compact metric spaces are proved.

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We let (X, d) be a complete metric space and let $B(X)$ be the set of all nonempty subsets of X . As in [1, 2], we define the function $\delta(A, B)$ with A and B in $B(X)$ by $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$. If A consists of a single point a we write $\delta(A, B) = \delta(a, B)$. If B also consists of single point b , we write $\delta(A, B) = \delta(a, B) = \delta(a, b) = d(a, b)$. It follows immediately that $\delta(A, B) = \delta(B, A) \geq 0$, and $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for all A, B in $B(X)$.

Now if $\{A_n : n = 1, 2, \dots\}$ is a sequence of sets in $B(X)$, we say that it converges to the set A in $B(X)$ if

- (i) each point $a \in A$ is the limit of some convergent sequence $\{a_n \in A_n : n = 1, 2, \dots\}$,
- (ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$ for $n > N$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ .

The set A is then said to be the limit of the sequence $\{A_n\}$.

The following lemma was proved [1].

LEMMA. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d) which converge to the bounded subsets A and B , respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Now, let F be a mapping of X into $B(X)$. We say that the mapping F is continuous at a point x if whenever $\{x_n\}$ is a sequence of points in X converging to x , the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is continuous mapping of X into $B(X)$ if F is continuous at each point x in X . We say that a point z in X is a fixed point of F if z is in Fz . If A is in $B(X)$, we define the set $FA = \bigcup_{a \in A} Fa$.

THEOREM 1. *Let (X, d_1) and (Y, d_2) be complete metrics spaces, let F be mapping of X into $B(Y)$ and let G be mapping of Y into $B(X)$ satisfying the inequalities*

$$\delta_1(GFx, GFx') \leq c \max\{d_1(x, x'), \delta_1(x, GFx), \delta_1(x', GFx'), \delta_2(Fx, Fx')\}, \quad (1)$$

$$\delta_2(FGy, FGy') \leq c \max \{d_2(y, y'), \delta_2(y, FGy), \delta_1(y', FGy'), \delta_1(Gy, Gy')\} \quad (2)$$

for all x, x' in X and y, y' , where $0 \leq c < 1$. If F is continuous, then GF has a unique fixed point z in X and FG has a unique fixed point w in Y .

PROOF. Let x_1 be an arbitrary point in X . Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y , respectively, as follows. Choose a point y_1 in Fx_1 and then a point x_2 in Gy_1 . In general, having chosen x_n in X and y_n in Y choose x_{n+1} in Gy_n and then y_{n+1} in Fx_{n+1} for $n = 1, 2, \dots$. Then,

$$\begin{aligned} d_1(x_{n+1}, x_{n+2}) &\leq \delta_1(GFx_n, GFx_{n+1}) \\ &\leq c \max \{d_1(x_n, x_{n+1}), \delta_1(x_n, GFx_n), \delta_1(x_{n+1}, GFx_n), \\ &\quad \delta_1(x_{n+1}, GFx_{n+1}), \delta_2(Fx_n, Fx_{n+1})\} \\ &\leq c \max \{\delta_1(GFx_{n-1}, GFx_n), \delta_1(GFx_n, GFx_{n+1}), \delta_2(Fx_n, Fx_{n+1})\} \\ &= c \max \{\delta_1(GFx_{n-1}, GFx_n), \delta_2(Fx_n, Fx_{n+1})\} \end{aligned} \quad (3)$$

and, similarly,

$$\begin{aligned} d_2(y_{n+1}, y_{n+2}) &\leq \delta_2(FGy_n, FGy_{n+1}) \\ &\leq c \max \{\delta_2(FGy_{n-1}, FGy_n), \delta_1(Gy_n, Gy_{n+1})\}. \end{aligned} \quad (4)$$

It follows that, for $r = 1, 2, \dots$,

$$\begin{aligned} d_1(x_{n+1}, x_{n+r+1}) &\leq \delta_1(GFx_n, GFx_{n+r}) \\ &\leq \delta_1(GFx_n, GFx_{n+1}) + \dots + \delta_1(GFx_{n+r-1}, GFx_{n+r}) \\ &\leq (c^n + c^{n+1} + \dots + c^{n+r-1})\delta_1(x_1, GFx_1) < \epsilon \end{aligned} \quad (5)$$

for n greater than some N , since $c < 1$. The sequence $\{x_n\}$ is, therefore, a Cauchy sequence in the complete metric space X and so has a limit z in X . Similarly, the sequence $\{y_n\}$ is a sequence in complete metric space Y and so has a limit w in Y .

Further

$$\begin{aligned} \delta_1(z, GFx_n) &\leq d_1(z, x_{m+1}) + \delta_1(x_{m+1}, GFx_n) \\ &\leq d_1(z, x_{m+1}) + \delta_1(GFx_m, GFx_n), \end{aligned} \quad (6)$$

since $x_{m+1} \in GFx_m$. Thus, on using inequality (5), we have

$$\delta_1(z, GFx_n) \leq d_1(z, x_{m+1}) + \epsilon \quad (7)$$

for $m, n \geq N$. Letting m tends to infinity it follows that

$$\delta_1(z, GFx_n) < \epsilon \quad (8)$$

for $n > N$ and so

$$\lim_{n \rightarrow \infty} GFx_n = \{z\} \quad (9)$$

since ϵ is arbitrary. Similarly,

$$\lim_{n \rightarrow \infty} FGy_n = \{w\} = \lim_{n \rightarrow \infty} Fx_n \quad (10)$$

since x_{n+1} is in $G\mathcal{Y}_n$. Using the continuity of F , we see that

$$\lim_{n \rightarrow \infty} Fx_n = Fz = \{w\}. \quad (11)$$

Using inequality (1), we now have

$$\delta_1(GFx_n, GFz) \leq c \max \{d_1(x_n, z), \delta_1(x_n, GFx_n), \delta_1(z, GFz), \delta_2(Fz, Fx_n)\}. \quad (12)$$

Letting n tends to infinity and using (9) and (11), we have

$$\delta_1(z, GFz) \leq c \delta_1(z, GFz). \quad (13)$$

Since $c < 1$, $\delta_1(z, GFz) = 0$ and, so, we must have $GFz = \{z\}$, proving that z is a fixed point of GF .

Further, using (11), we have

$$FGw = FGFz = Fz = w, \quad (14)$$

proving that w is a fixed point of FG .

Now suppose that GF has a second fixed point z' . Then using inequalities (1) and (2), we have

$$\begin{aligned} \delta_1(z', GFz') &\leq \delta_1(GFz', GFz') \\ &\leq c \max \{d_1(z', z'), \delta_1(z', GFz'), \delta_2(Fz', Fz')\} \\ &= c \delta_2(Fz', Fz') \leq c \delta_2(Fz', FGFz') \leq c \delta_2(FGFz', FGFz') \\ &\leq c^2 \max \{\delta_2(Fz', Fz'), \delta_2(Fz', FGFz'), \delta_1(GFz', FGFz')\} \\ &= c^2 \delta_2(GFz', GFz') \end{aligned} \quad (15)$$

and so Fz' is a singleton and $GFz' = \{z'\}$, since $c < 1$. Thus,

$$\begin{aligned} d_1(z, z') &= \delta_1(GFz, GFz') \\ &\leq c \max \{d_1(z, z'), \delta_1(z, GFz), \delta_1(z', GFz'), \delta_2(Fz, Fz')\} \\ &= c d_2(Fz, Fz'). \end{aligned} \quad (16)$$

But

$$\begin{aligned} d_2(Fz, Fz') &\leq \delta_2(FGFz, FGFz') \\ &\leq c \max \{\delta_2(Fz, Fz'), \delta_2(Fz, FGFz), \delta_2(Fz', FGFz'), \delta_1(GFz, GFz')\} \\ &= c \max \{d_2(Fz, Fz'), d_2(Fz, Fz), d_2(Fz', Fz'), d_1(z, z')\} \\ &= c d_1(z, z') \end{aligned} \quad (17)$$

and so

$$d_1(z, z') \leq c^2 d_1(z, z'). \quad (18)$$

Since $c < 1$, the uniqueness of z follows.

Similarly, w is the unique fixed point of FG . This completes the proof of the theorem. $\square$

If we let F be a single valued mapping T of X into Y and G be a single valued mapping of Y into X , we obtain the following result given in [3].

COROLLARY 1. *Let (X, d_1) and (Y, d_2) be complete metric spaces. If T is a continuous mapping of X into Y , and S is a mapping of Y into X satisfying the inequalities*

$$\begin{aligned} d_1(STx, STx') &\leq c \max \{d_1(x, x'), d_1(x, STx), d_1(x', STx'), d_2(Tx, Tx')\}, \\ d_2(STy, STy') &\leq c \max \{d_2(y, y'), d_2(y, TSy), d_2(y', TSy'), d_1(Sy, Sy')\} \end{aligned} \quad (19)$$

for all x, x' in X and y, y' in Y , where $0 \leq c < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y . Further $Tz = w$ and $Sw = z$.

THEOREM 2. *Let (X, d_1) and (Y, d_2) be compact metric spaces. If F is a continuous mapping of X into $B(Y)$, and G is a continuous mapping of Y into $B(X)$ satisfying the inequalities*

$$\begin{aligned} \delta_1(GFx, GFx') &< \max \{d_1(x, x'), \delta_1(x, GFx), \delta_1(x', GFx'), \delta_2(Fx, Fx')\}, \\ \delta_2(FGy, FGy') &< \max \{d_2(y, y'), \delta_2(y', FGy), \delta_2(y', FGy'), \delta_1(Gy, Gy')\} \end{aligned} \quad (20)$$

for all x, x' in X and y, y' in Y for which the right-hand sides of the inequalities are positive. Then FG has a unique fixed point z in X and GF has a unique fixed point w in Y . Further $FGz = \{z\}$ and $GFw = \{w\}$.

PROOF. Let us denote the right-hand side of inequalities (20) by $h(x, x')$ and $k(y, y')$, respectively. First of all suppose that $h(x, x') \neq 0$ for all $x, x' \in X$ and $k(y, y') \neq 0$ for all $y, y' \in Y$. Define the real-valued function $f(x, x')$ on X^2 by

$$f(x, x') = \frac{\delta_1(GFx, GFx')}{h(x, x')}. \quad (21)$$

Then if $\{(x_n, x'_n)\}$ is an arbitrary sequence in X^2 converging to (x, x') , it follows from the lemma and the continuity of F and G the sequence $\{f(x_n, x'_n)\}$ converges to $f(x, x')$. The function f is therefore a continuous function defined on the compact metric space X^2 and so achieves its maximum value c_1 . Because of inequality (9), $c_1 < 1$ and so

$$\delta_1(GFx, GFx') \leq c_1 \max \{d_1(x, x'), \delta_1(x, GFx), \delta_1(x', GFx'), \delta_2(Fx, Fx')\} \quad (22)$$

for all x, x' in X .

Similarly, there exists $c_2 < 1$ such that

$$\delta_2(FGy, FGy') \leq c_2 \max \{d_2(y, y'), \delta_2(y, FGy), \delta_2(y', FGy'), \delta_1(Gy, Gy')\} \quad (23)$$

for all y, y' in Y . It follows that the conditions of Theorem 2 are satisfied with $c = \max\{c_1, c_2\}$ and, so, once again there exist z in X and w in Y such that $GFz = \{z\}$ and $FGw = \{w\}$.

Now, suppose that $h(x, x') = 0$ for some x, x' in X . Then $Gfx = GFx' = \{x\} = \{x'\}$ is a singleton $\{z\}$ and then Fz is a singleton $\{w\}$. It follows that z is a fixed point of

GF and $GFz = \{z\}$. Further,

$$FGw = FGFz = Fz = \{w\} \quad (24)$$

and so w is a fixed point of FG .

It follows similarly that if $k(y, y') = 0$ for some y, y' in Y , then again GF has a fixed point z and FG has a fixed point w .

Now let us suppose that GF has a second fixed point z' in X so that z' is in GFz' . Then on using inequalities (20), we have, on assuming that $\delta_2(Fz', Fz') \neq 0$,

$$\begin{aligned} \delta_1(z', GFz') &\leq \delta_1(GFz', GFz') \\ &< \max \{ \delta_1(z', z'), \delta_1(z', GFz'), \delta_2(Fz', Fz') \} \\ &= \delta_2(Fz', Fz') \leq \delta_2(Fz', FGFz') \leq \delta_2(FGFz', FGFz') \\ &< \max \{ \delta_2(Fz', Fz'), \delta_2(Fz', FGFz'), \delta_1(GFz', FGFz') \} \\ &= c^2 \delta_2(GFz', GFz') \end{aligned} \quad (25)$$

a contradiction and so Fz' is a singleton and $GFz' = \{z'\}$. Thus, if $z \neq z'$

$$\begin{aligned} d_1(z, z') &= \delta_1(GFz, GFz') \\ &< \max \{ d_1(z, z'), \delta_1(z, GFz), \delta_1(z', GFz'), \delta_2(Fz, Fz') \} \\ &= d_2(Fz, Fz'). \end{aligned} \quad (26)$$

But if $Fz \neq Fz'$, we have

$$\begin{aligned} d_2(Fz, Fz') &\leq \delta_2(FGFz, FGFz') \\ &< \max \{ \delta_2(Fz, Fz'), \delta_2(Fz, FGFz), \delta_2(Fz', FGFz'), \delta_1(GFz, GFz') \} \\ &= \max \{ \delta_2(Fz, Fz'), d_2(Fz, Fz), d_2(Fz', Fz'), d_1(z, z') \} \\ &= d_1(z, z') \end{aligned} \quad (27)$$

and so

$$d_1(z, z') < d_1(z, z'), \quad (28)$$

a contradiction. The uniqueness of z follows.

Similarly, w is the unique fixed point of FG . This completes the proof of the theorem. $\square$

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for all x, x' in X and y, y' in Y for which the right-hand sides of the inequalities are positive, then ST has a fixed point z in X and TS has a unique fixed point w in Y . Further, $Tz = w$ and $Sw = z$.

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