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# THE MATCHING POLYNOMIAL OF A DISTANCE-REGULAR GRAPH

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ABSTRACT. A distance-regular graph of diameter $d$ has $2d$ intersection numbers that determine many properties of graph (e.g., its spectrum). We show that the first six coefficients of the matching polynomial of a distance-regular graph can also be determined from its intersection array, and that this is the maximum number of coefficients so determined. Also, the converse is true for distance-regular graphs of small diameter—that is, the intersection array of a distance-regular graph of diameter 3 or less can be determined from the matching polynomial of the graph.

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**1. Introduction.** Distance-regular graphs are highly regular combinatorial structures that often occur in connection with other areas of combinatorics (e.g., designs and finite geometries) and many of their properties can be determined from their intersection numbers. These properties include the eigenvalues of the graph, and their multiplicities, and hence, we can determine the characteristic polynomial of a distance-regular graph from knowledge of its intersection numbers. It is also known that the characteristic polynomial of any graph can be determined by computing the circuit polynomial and converting it to a polynomial in a single variable via a specific set of substitutions.

Since the intersection array and the circuit polynomial of a distance-regular graph both determine its characteristic polynomial, it was natural to investigate the relationship between the circuit polynomial and the intersection array for distance-regular graphs. Specifically, could the circuit polynomial be determined from the intersection array? We show that the answer to this question is no, even though a portion of the circuit polynomial (which is also part of the matching polynomial) can be computed from just the intersection array.

We concerned here with the matching polynomial of a distance-regular graph, which is a constituent of the circuit polynomial. The initial portion of the matching polynomials (and other graph polynomials) of many common regular graphs have been computed (e.g., [7, 8, 9]). In the cases where these regular graphs are distance-regular (complete graphs, circuit, complete bipartite graphs, hypercubes), our results serve to generalize and unify the determination of the initial coefficients of the matching polynomials. Furthermore, the matching polynomial is also a constituent of many other graph polynomials (in addition to the circuit polynomial), so these results also apply to other more specific polynomials.

**2. Graph polynomials.** The matching polynomial is an example of a general graph polynomial, which we now describe. The first element in the construction of a graph polynomial is a family of graphs, $F$, such as all trees, or all circuits. Typically such a family is infinite, and often includes a single vertex and a single edge as members. To each member of this family a weight is assigned. Often this weight is an indeterminate, which is subscripted by either the number of vertices or the number of edges in the graph. Having chosen a family $F$ and a weighting scheme, we compute the *$F$-polynomial* of a graph $G$ by first finding the spanning subgraphs of $G$ where each component is an element of $F$. Such a spanning subgraph is called an *$F$-cover*. For each cover, take the product of the weights of the components and then sum these terms over all the $F$-covers of the graph. The resulting polynomial is the $F$-polynomial of $G$. Throughout this paper, we consider only $F$-polynomials constructed by assigning the indeterminate $w_i$ to a component with $i$ vertices. For more on the general properties of $F$-polynomials, see [6].

Presently, we are interested in the matching polynomial of a graph. We take $F$ to be the family consisting of just a vertex and an edge. In this case, a cover will consist of disjoint edges and isolated vertices—a matching in the graph. The resulting *matching polynomial* has terms of the form $cw_1^{n-2m}w_2^m$, where $n$ is the number of vertices, and $c$ is the number of matchings in $G$ that have $m$ edges. Thus, finding the matching polynomial of a graph is equivalent to finding the number of $m$-matchings in the graph, for all $m$.

Many of the families used to construct interesting $F$-polynomials include a vertex and an edge, and therefore all of the matchings of the graph are created as $F$-covers. The terms of the form $cw_1^{n-2m}w_2^m$ in the $F$-polynomial then coincide with the matching polynomial itself, in other words, the matching polynomial is a subpolynomial of the $F$-polynomial. For example, the circuit polynomial is formed by taking $F$ to be the set of all circuits, with a vertex and an edge viewed as degenerate circuits on 1 and 2 vertices, respectively. So, the matching polynomial is a subpolynomial of the circuit polynomial. As mentioned in the introduction, the characteristic polynomial of a graph can be determined from the circuit polynomial. The characteristic polynomial in the single variable $\lambda$ is obtained by making the following substitutions into the circuit polynomial [6]: $w_1 = \lambda$, $w_2 = -1$, and $w_m = -2$, $m > 2$.

**3. Matching polynomials.** In order to find the initial portion of the matching polynomial of a distance-regular graph, we use some results from [1, 2] that give expressions for the number of certain subgraphs present in a regular graph. The coefficients of the matching polynomial are equal to the number of matchings in the graph that have a specified number of edges. For an arbitrary regular graph, we give expressions for the number of such matchings that have five or fewer edges. More details and proofs of these results can be found in [1, 2].

We adopt the following notation: For a regular graph $G$, let $n$ be the number of vertices, and $r$ be the degree. We also require the number of certain types of subgraphs in $G$ (it is important to note that these are not vertex induced subgraphs, but they are simply graphs whose edge and vertex sets are subsets of those of $G$). Let $M_i$ be the number of matchings on $i$ edges, that is, subgraphs with $i$ edges and $2i$ vertices (each

of degree 1). Let $T, S$, and $P$ be the number of subgraphs, respectively, that are 3-cycles (triangles), 4-cycles (squares), and 5-cycles (pentagons). Finally, let $N$ be the number of subgraphs that consist of a triangle together with a single edge joining a fourth vertex to one vertex of the triangle, and let $H$ be the number of subgraphs equal to a complete graph on 4 vertices with one edge deleted.

**Theorem 3.1.** *Suppose that $G$ is a regular graph of degree $r$ on $n$ vertices. Then, using the notation given above, we have*

$$N = 3(r-2)T, \tag{3.1}$$

$$M_0 = 1, \tag{3.2}$$

$$M_1 = \frac{nr}{2}, \tag{3.3}$$

$$M_2 = \frac{nr(2-4r+nr)}{8}, \tag{3.4}$$

$$M_3 = \frac{nr(16-48r+6nr+40r^2-12nr^2+n^2r^2)}{48} - T, \tag{3.5}$$

$$\begin{aligned} M_4 = \frac{nr}{384}&(240-960r+76nr+1344r^2-240nr^2+12n^2r^2-672r^3 \\ &+208nr^3-24n^2r^3+n^3r^3) + \left(-6+6r-\frac{nr}{2}\right)T+S, \end{aligned} \tag{3.6}$$

$$\begin{aligned} M_5 = \frac{nr}{3840}&(5376-26880r+1520nr+51840r^2-6400nr^2 \\ &+220n^2r^2-46080r^3+9440nr^3-720n^2r^3+20n^3r^3 \\ &+16128r^4-4960nr^4+640n^2r^4-40n^3r^4+n^4r^4) \\ -\frac{1}{8}&(216-432r+26nr+216r^2-28nr^2+n^2r^2)T \\ +&\left(8-8r+\frac{nr}{2}\right)S-P+4H. \end{aligned} \tag{3.7}$$

**Proof.** Results from [1, 2] describe how to construct a set of linear equations which when solved yield these expressions (in addition to others that specify the number of other types of subgraphs). The original linear equations were generated by a PASCAL program, and the solutions were found using the symbolic manipulation capabilities of the program *Mathematica*. □

**4. Distance-regular graphs.** Loosely speaking, a distance-regular graph has much combinatorial regularity, which in turn implies some amazing algebraic properties. Examples include complete graphs, complete bipartite graphs, hypercubes, Petersen's graph and some of its generalizations, line graphs of some distance-regular graphs, and graphs related to other incidence structures such as designs and finite geometries. A good introduction to distance-regular graphs can be found in [3], and a more advanced treatment is given in [4]. Here, we list some pertinent facts that are needed later.

Let $\partial(u,v)$ denote the distance between two vertices $u$ and $v$. Then, given vertices $u$ and $v$, at a distance $i$ apart in a graph, the *intersection number* of $(u,v)$ is defined as

$$s_{ijk}(u,v) = \left|\{w \mid \partial(u,w) = j,\ \partial(w,v) = k\}\right|. \tag{4.1}$$

A graph is *distance-regular* if the intersection numbers depend on the choice of $i, j$, and $k$, but not on the choice of the particular pair of vertices, $u$ and $v$, that are a distance $i$ apart. A distance-regular graph is usually described by referencing the following special cases of the intersection numbers (where the subscripts make sense),

$$a_i = s_{i,i,1}, \qquad b_i = s_{i,i+1,1}, \qquad c_i = s_{i,i-1,1}. \tag{4.2}$$

It can be shown that knowledge of these numbers is sufficient to compute all of the intersection numbers of a distance-regular graph. A distance-regular graph must be regular, and if the degree is $b_0 = r$, then $a_i + b_i + c_i = r$. Thus, if the graph has diameter $d$, the intersection numbers are determined by the *intersection array*, $\{b_0, b_1, \ldots, b_{d-1}; c_1 = 1, c_2, \ldots, c_d\}$.

It can be shown that the intersection array of a distance-regular graph is sufficient to determine the characteristic polynomial of the graph (see [3]).

**Theorem 4.1.** *The first six terms of the matching polynomial of a distance-regular graph are determined by the intersection array of the graph.*

**Proof.** Suppose that $G$ is a distance-regular graph with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1 = 1, c_2, \ldots, c_d\}$. We take each term of the matching polynomial in turn.

**First term.** Since there is only one matching with no edges, the first term of the matching polynomial has the form $w_1^n$, where $n$ is the number of vertices in $G$. Let $G_i(v)$ denote the set of vertices at distance $i$ from vertex $v$, and let $k_i = |G_i(v)|$. Then, by counting the edges joining the vertices in $G_{i-1}$ to the vertices in $G_i$, we find that $b_{i-1}k_{i-1} = c_i k_i$. Repeated use of this equation, together with the initial condition $k_0 = 1$, allows us to find $n$ via $n = \sum_{i=o}^{d} k_i$, and, thus, we can obtain the first term from the intersection array.

**Second term.** The number of matchings that have a single edge is equal to the number of edges in the graph. If we let $r$ denote the degree of the vertices in the graph, then the coefficient of $w_1^{n-2}w_2$ is $nr/2$ (3.3). However, $r = b_0$, so we are able to obtain this coefficient from the intersection array.

**Third term.** The coefficient of $w_1^{n-4}w_2^2$ is given by (3.4), and depends solely on $n$ and $r$ which we have already seen, can be determined from the intersection array.

**Fourth term.** The coefficient of $w_1^{n-6}w_2^3$ is given by (3.5) and depends on $n, r$ and the number of triangles $T$ in the graph. How many triangles does a distance-regular graph have?

Let $v$ be a vertex of $G$, and let $T_v$ be the number of triangles that pass through $v$. The edge opposite to $v$ in any such triangle joins two vertices of $G_1(v)$, and any edge joining two vertices of $G_1(v)$ yields a triangle that has $v$ as a vertex. Now, $G_1(v)$ induces a subgraph that is regular of degree $a_1$ on $k_1$ vertices and, therefore, has $a_1k_1/2$ edges. Since each of these edges corresponds to a triangle, we have $T_v = a_1k_1/2$. Then, if we sum over all the vertices of $G$, we count each triangle three times, once for each vertex. Also, $k_1 = r$, so

$$3T = \sum_v T_v = \frac{nk_1a_1}{2} \Longrightarrow T = \frac{nra_1}{6}. \tag{4.3}$$

Thus, we can obtain the number of triangles from the intersection array. Together with $n$ and $r$, we can then find the fourth coefficient.

**Fifth term.** The coefficient of $w_1^{n-8}w_2^4$ is given by (3.6) and depends on $n, r, T$, and $S$.

We count the number of squares that have $v$ as a vertex. Let $v$ be a vertex of $G$, and let $S_v$ be the number of squares that contain $v$ as a vertex. Let $w$ be the vertex opposite to $v$ in a square, that is $\partial(v,w) = 2$ in the square. However, the distance between $v$ and $w$ may be 1 or 2 in $G$, so we consider two cases.

**Case 1.** Suppose that $\partial(v,w) = 1$. The vertex $w$ is adjacent to $a_1$ vertices in $G_1(v)$ and, for each pair of these vertices, there corresponds a square through $v$ with $w$ as the opposite vertex. Thus, we get $\binom{a_1}{2}$ squares through $w$ and there are $k_1$ choices for $w$, yielding a total of $k_1\binom{a_1}{2}$ squares in this case.

**Case 2.** Suppose that $\partial(v,w) = 2$. The vertex $w$ is now an element of $G_2(v)$ and has $c_2$ neighbors in $G_1(v)$. For each pair of these neighbors, there corresponds a square through $v$ with $w$ as the opposite vertex. Thus, we get $\binom{c_1}{2}$ squares through $w$ and there are $k_2$ choices for $w$, yielding a total of $k_2\binom{c_2}{2}$ squares in this case.

Combining these two cases and using $k_2c_2 = k_1b_1$ and $k_1 = r$, we get

$$\begin{aligned} S_v &= k_1\binom{a_1}{2} + k_2\binom{c_2}{2} \\ &= \frac{1}{2}\big(k_1a_1(a_1-1) + k_2c_2(c_2-1)\big) \\ &= \frac{1}{2}\big(ra_1(a_1-1) + k_1b_1(c_2-1)\big) \\ &= \frac{r}{2}\big(a_1(a_1-1) + b_1(c_2-1)\big). \end{aligned} \tag{4.4}$$

If we sum $S_v$ over all the vertices, we can count each square 4 times, so

$$\begin{aligned} 4S &= \sum_v S_v = \frac{nr}{2}\big(a_1(a_1-1) + b_1(c_2-1)\big), \\ S &= \frac{nr}{8}\big(a_1(a_1-1) + b_1(c_2-1)\big). \end{aligned} \tag{4.5}$$

Now, since we see that the number of squares can be determined from the intersection array, we see that we can also determine the fifth coefficient.

**Sixth term.** The coefficient of $w_1^{n-10}w_2^5$ is given by (3.7) and depends on $n, r, T, S$, $P$, and $H$.

Fix a vertex $v$, and let $H_v$ be the number of graphs counted in $H$ that have vertex $v$ as one of the vertices of degree 3. The other three vertices must be in $G_1(v)$ and induce a subgraph in $G_1(v)$ which is a path of length 2. There are $k_1$ ways to choose the central vertex of the path and $\binom{a_1}{2}$ ways to choose the adjacent endpoints. To each such triple of vertices from $G_1(v)$ corresponds a graph counted in $H_v$, so $H_v = k_1\binom{a_1}{2}$. If we sum $H_v$ over all the vertices of $G$, then we can count each graph twice, so

$$2H = \sum_v H_v = nk_1 \binom{a_1}{2} \Longrightarrow H = \frac{nr}{4} a_1(a_1 - 1). \tag{4.6}$$

Fix a vertex $v$, and let $P_v$ be the number of graphs counted in $P$ which have $v$ as a vertex. To obtain an expression for $P_v$, we consider the two vertices $x$ and $y$, which are a distance 2 from $v$ in the pentagon. These two vertices may be a distance 1 or 2 from $v$ in $G$, which gives rise to the following three cases.

**Case 1.** Suppose that $\partial(v,x) = \partial(v,y) = 2$. In this case, $x$ and $y$ are adjacent vertices in $G_2(v)$. Then $G_2(v)$ induces a regular graph of degree $a_2$ on $k_2$ vertices and, thus, has $k_2a_2/2$ edges. Each such edge determines a pair of vertices $x$ and $y$. Now we must connect both $x$ and $y$ to vertices $w$ and $z$, respectively, in $G_1(v)$($w$ and $z$ may not be distinct). In each case, this can be done in $c_2$ ways. So, we can create $c_2^2k_2a_2/2$ apparent pentagons, except when $w = z$. In this case, we have a graph that is composed of a triangle, with an additional vertex of degree 1 adjacent to one of the vertices of the triangle—a graph of the type counted by $N$. Here, $v$ is the vertex of degree 1, and $x$ and $y$ are the two vertices of degree 2. Since $x$ and $y$ are a distance 2 from $v$ in $G$, we denote the number of such graphs as $N_{v,22}$.

**Case 2.** Suppose that $\partial(v,x) = \partial(v,y) = 1$. In this case, $x$ and $y$ are adjacent vertices in $G_1(v)$. Then, $G_1(v)$ induces a regular graph of degree $a_1$ on $k_1$ vertices and, thus, has $k_1a_1/2$ edges. Each such edge determines a pair of vertices $x$ and $y$. Now we must connect both $x$ and $y$ to new vertices $w$ and $z$, respectively, also, in $G_1(v)$. In each case, this can be done in $a_1 - 1$ ways. So, we can create $(a_1-1)^2k_1a_1/2$ apparent pentagons, except that, again, we have not prevented the possibility that we have chosen $w = z$. In the case that $w = z$, we again obtain graphs counted by $N$ that we count as $N_{v,11}$ since the two vertices of degree 2 are a distance 1 from $v$ in $G$.

**Case 3.** Suppose that $\partial(v,x) = 1, \partial(v,y) = 2$. We can assume, without loss of generality, that $y$ is the vertex in $G_2(v)$—there are $k_2$ ways for this to occur. There are $c_2$ vertices in $G_1(v)$ which are adjacent to $y$, and which could be $x$. Now we need to choose two more vertices in $G_1(v)$. First, $z$ should be adjacent to $y$ and should not be equal to $x$. This can be done in $c_2 - 1$ ways. Second, $w$ should be adjacent to $x$, which can be done in $a_1$ ways. So, it appears that there are $k_2c_2(c_2-1)a_1$ pentagons, but we have not ruled out the possibility that $w = z$. In the case that $w = z$, we obtain graphs that we count as $N_{v,12}$ since the two vertices of degree 2 are at distances 1 and 2 from $v$ in $G$.

To consolidate these three cases, notice that if $N_v$ is the number of graphs counted in $N$ that have $v$ as the single vertex of degree 1, then $N_v = N_{v,11} + N_{v,12} + V_{v,22}$. Then, we have

$$\begin{aligned} P_v + N_{v,11} + N_{v,12} + N_{v,22} &= \frac{(a_1-1)^2k_1a_1}{2} + k_2c_2(c_2-1)a_1 + \frac{c_2^2k_2a_2}{2}, \\ P_v + N_v &= \frac{(a_1-1)^2k_1a_1}{2} + k_1b_1(c_2-1)a_1 + \frac{k_1b_1c_2a_2}{2}, \\ P_v + N_v &= r\left(\frac{a_1(a_1-1)^2}{2} + a_1b_1(c_2-1) + \frac{b_1a_2c_2}{2}\right). \end{aligned} \tag{4.7}$$

If we sum this expression over all the vertices $v$ in $G$, we count each pentagon 5 times

and each graph counted in $N$ just once. Using (3.1), we have

$$\begin{aligned} 5P+N &= \sum_v p_v + N_v = nr\left(\frac{a_1(a_1-1)^2}{2} + a_1b_1(c_2-1) + \frac{b_1a_2c_2}{2}\right), \\ 5P &= nr\left(\frac{a_1(a_1-1)^2}{2} + a_1b_1(c_2-1) + \frac{b_1a_2c_2}{2}\right) - \frac{3(r-2)nra_1}{6}, \\ P &= \frac{nr}{10}(a_1(a_1-1)^2 + 2a_1b_1(c_2-1) + b_1a_2c_2 - (r-2)a_1). \end{aligned} \tag{4.8}$$

Since both $H$ and $P$ can be determined from the intersection array, we can also determine the sixth coefficient from the intersection array. □

The following example illustrates that this result is the best possible. We begin by considering the following pair of distance-regular graphs, which are not isomorphic, yet have the same set of intersection numbers. The first graph is the Hamming scheme, $H(2,4)$, which has vertices that are strings of length 2 over an alphabet with 4 letters. Vertices are adjacent if the corresponding strings are different in just one position. The resulting graph on 16 vertices is regular of degree 6 and has the intersection array $\{6,3;1,2\}$. In [5], Egawa describes a distance-regular graph found by Shrikhande [10] that has the same intersection array as $H(2,4)$, yet is not isomorphic to $H(2,4)$.

Since these two graphs have identical intersection arrays, they have identical characteristic polynomials. However, their circuit polynomials are not equal. This can be seen quickly by comparing their matching polynomials, which are sub-polynomials of their respective circuit polynomials. The matching polynomials are

$$\begin{aligned} H(2,4):\quad & w_1^{16} + 48w_1^{14}w_2 + 888w_1^{12}w_2^2 + 8064w_1^{10}w_2^3 + 37944w_1^8w_2^4 \\ & + 89856w_1^6w_2^5 + 96000w_1^4w_2^6 + 35712w_1^2w_2^7 + 2016w_2^8, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \text{Shrikhande:}\quad & w_1^{16} + 48w_1^{14}w_2 + 888w_1^{12}w_2^2 + 8064w_1^{10}w_2^3 + 37944w_1^8w_2^4 \\ & + 89856w_1^6w_2^5 + 95872w_1^4w_2^6 + 35328w_1^2w_2^7 + 1920w_2^8. \end{aligned} \tag{4.10}$$

Of course, the first six coefficients of these two polynomials are equal, yet they are different in their seventh coefficient. Since their matching polynomials differ, their circuit polynomials cannot be equal and, thus, it is clear that we cannot determine the circuit polynomial (or for that matter, the matching polynomial) of an arbitrary distance-regular graph by knowing just the intersection array.

As an exercise, we can apply the results contained in the proof above to the intersection array of the Hamming scheme and Shrikhande graph and obtain $n = 16$, $r = 6$, $T = 32$, $S = 60$, $H = 48$, $P = 288$. Then, using these values in (3.2), (3.3), (3.4), (3.5), (3.6), and (3.7) yield the first six coefficients of the matching polynomials, as given in (4.9) and (4.10).

An expression analogous to (3.2), (3.3), (3.4), (3.5), (3.6), and (3.7) can be found for $M_6$, the seventh coefficient of the matching polynomial. It depends on $n, r, T, S, P, H$, and the numbers of each of six other subgraphs. These six subgraphs are the subgraphs with six edges and no vertices of degree 1, one of which is the complete graph on

four vertices. The number of subgraphs of a distance-regular graph that are complete graphs on four vertices cannot be determined from the intersection array. This can be demonstrated in the previous example since the Hamming graph has eight subgraphs that are complete on four vertices, while the Shrikhande graph has none. This partially explains the discrepancy in the seventh coefficient of the matching polynomials of these two graphs.

The previous theorem shows how to determine a portion of the matching polynomial of a distance-regular graph from its intersection array—the converse is possible for graphs of sufficiently small diameter.

**Theorem 4.2.** *Suppose that $G$ is a distance regular graph of diameter 3 or less. Then its intersection array can be found from its matching polynomial.*

**Proof.** We give the proof for graphs of diameter 3. It should be obvious how to shorten the proof for the cases of smaller diameter. Suppose that we have the matching polynomial of a distance-regular graph of diameter 3.

The first term of the matching polynomial, $w_1^n$, allows us to determine the number of vertices in the graph, $n$. The coefficient of the second term is $nr/2$, and together with $n$, this is sufficient to determine the degree of the graph $r$. Because $b_0$ is the degree of the graph, we have the first element of the intersection array.

The coefficient of the fourth term depends on $n,r$, and the number of triangles, $T$ (3.5). Thus, we can determine the number of triangles in the graph. In turn, $T$ depends on $n,r$, and $a_1$ (4.3). Because we know $n$ and $r$ and the dependence on $a_1$ is linear, we can then find $a_1$. Because $c_1 = 1$ for any distance-regular graph, and $r = a_1 + b_1 + c_1$, we can also determine $b_1$.

The coefficient of the fifth term depends on $n,r,T$, and the number of squares, $S$ (3.6). Thus, we can determine the number of squares in the graph. In turn, $S$ depends on $n,r,a_1,b_1$, and $c_2$ (4.5). Because we know $n,r,a_1$, and $b_1$ and since the dependence on $c_2$ is linear, we can then find $c_2$.

We can compute $H$ with the information at hand since we need only $n,r$, and $a_1$ (4.6). The coefficient of the sixth term depends on $n,r,T,S,H$, and the number of pentagons, $P$ (3.6). Thus, we can determine the number of pentagons in the graph. In turn, $P$ depends on $n,r,a_1,b_1,c_2$, and $a_2$ (4.8). Because we know $n,r,a_1,b_1,c_2$ and since the dependence on $a_2$ is linear, we can then find $a_2$.

With the current collection of intersection numbers, we can in turn find $b_2,c_3$, and $a_3$ with the following sequence of equations:

$$\begin{aligned} r &= a_2 + b_2 + c_2, & k_1 &= r, \\ k_1 b_1 &= k_2 c_2, & n &= 1 + k_1 + k_2 + k_3, \\ k_2 b_2 &= k_3 c_3, & r &= a_3 + c_3. \end{aligned} \tag{4.11}$$

Thus, the matching polynomial of a distance-regular graph of diameter 3 determines its entire intersection array. □

Recall that the intersection array of a distance-regular graph determines the characteristic polynomial of the graph and, thus, determines the eigenvalues and their multiplicities, for the graph. In the case of small diameter distance-regular graphs, because

the matching polynomial determines the intersection array, it also determines the eigenvalues and their multiplicities, for the graph. So, the first six coefficients of the matching polynomial carry enough information about small diameter distance-regular graphs to determine the entire spectrum.

In summary, we have seen that while both the intersection array and the circuit polynomial determine the spectrum of a distance-regular graph, in general, they are independent of each other. However, the initial portion of the matching polynomial (and hence, also the initial portion of the characteristic polynomial) of a distance-regular graph can be found from the intersection array of a distance-regular graph. Thus, for those classes of regular graphs that are also distance-regular, we can easily calculate the initial portion of the matching polynomial and some initial terms of other graph polynomials that contain the matching polynomial as a sub-polynomial.

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