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STATISTICAL LIMIT POINT THEOREMS

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ABSTRACT. It is known that given a regular matrix A and a bounded sequence x there is a subsequence (respectively, rearrangement, stretching) y of x such that the set of limit points of Ay includes the set of limit points of x. Using the notion of a statistical limit point, we establish statistical convergence analogues to these results by proving that every complex number sequence x has a subsequence (respectively, rearrangement, stretching) y such that every limit point of x is a statistical limit point of y. We then extend our results to the more general A-statistical convergence, in which A is an arbitrary nonnegative matrix.

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1. Introduction. In [2, 3] Buck characterized convergence by proving that if x is a nonconvergent sequence, then no regular matrix can sum every subsequence of x. This result was extended by Agnew [1] who showed that given a regular matrix A and a bounded sequence x, there is a subsequence y of x such that the set of limit points of Ay includes the set of limit points of x. Analogues to these results were given by Dawson [6] and Fridy [9] by replacing subsequence with stretching and rearrangement, respectively. In [17], statistical convergence analogues and A-statistical convergence analogues to Buck's theorem and its variants are given. Now we generalize the constructions in [17], providing statistical convergence analogues and A-statistical convergence analogues to Agnew's theorem and its variants.

If K is a subset of the natural numbers $\mathbb{N}$, let K_n denote the set $\{k \le n : k \in K\}$ and $|K_n|$ denote the cardinality of K_n. The *natural* or *asymptotic* density of K (see [13, Chapter 11]) is given by $\delta(K) = \lim_n (1/n)|K_n|$, if the limit exists. A complex number sequence x is said to be *statistically convergent to L* if for every positive ε,

$$\delta\left(\{k : |x_k - L| \ge \varepsilon\}\right) = 0. \tag{1.1}$$

In this case, we write st-lim$x = L$. This notion was first introduced by Fast [7] for real sequences and has since been studied as a regular summability method by several authors (cf. [5, 10, 14]). Using natural density, Fridy [11] defined an analogue to the notion of a limit point of a sequence x. A subsequence $y = \{x\}_K$ of x is said to be *nonthin* if K does not have natural density zero, and the number λ is said to be a *statistical limit point* of x if there exists a nonthin subsequence of x that converges to λ.

Natural density was generalized by Freedman and Sember [8] by replacing C_1 with an arbitrary nonnegative regular matrix A. Thus, if K is a subset of $\mathbb{N}$, then the A-density of K is given by

$$\delta_A(K) = \lim_{n\to\infty} \sum_{k\in K} a_{n,k} \tag{1.2}$$

if the limit exists. This notion was used by Kolk in [12] to extend statistical convergence as follows. A complex number sequence x is said to be *A-statistically convergent to L* if, for every positive ε,

$$\delta_A(\{k : |x_k - L| \geq \varepsilon\}) = 0. \tag{1.3}$$

In this case, we write $\mathrm{st}_A\text{-}\lim x = L$. Connor and Kline [4] replaced natural density with A-density in Fridy's definition of a statistical limit point. So a subsequence $y = \{x\}_K$ of the sequence x is called *A-nonthin* if K does not have A-density zero and the number λ is called an *A-statistical limit point* of x if there is an A-nonthin subsequence of x that converges to λ.

Before we can state our main results, we must give the following two definitions.

Definition 1.1. A sequence z is called a *rearrangement* of the sequence x provided that there is a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that for each k, $z_k = x_{\pi(k)}$.

Remark 1.2. The word "permutation" is reserved for the reordering of a finite sequence.

Definition 1.3 (Dawson [6]). Let $\{m(p)\}_{p=0}^{\infty}$ be an increasing sequence of integers such that $m(0) = 1$. We call the sequence w the *stretching of x induced by* $\{m(p)\}$ provided $w_q = x_p$, whenever $m(p-1) \leq q < m(p)$.

Remark 1.4. The sequence w has also been called a *dilution* of x by Sledd [15].

2. Statistical limit point theorems.

In [1] Agnew proved that, given a regular matrix A and a bounded sequence x, there is a subsequence y of x such that the set of limit points of Ay includes the set of limit points of x. If x is bounded but divergent, it has at least two distinct limit points and, so, Ay also has at least those same two limit points. Therefore y is not A-summable and Buck's theorem [2, 3] (for bounded sequences) follows from Agnew's theorem.

It is shown in [10, Theorem 1] that a sequence x is statistically convergent if and only if x is a sequence for which there exists a *convergent* sequence y such that $x_k = y_k$ for almost all k, that is, for every k in a set K with $\delta(K) = 1$. This implies that if x is statistically convergent to λ then the set of statistical limit points is the singleton set $\{\lambda\}$. In the bounded cases of [16, Theorems 2.1, respectively 2.2, 2.3], we created a subsequence (respectively, rearrangement, stretching) that had two distinct statistical limit points and was therefore not statistically convergent. In this section, we generalize those constructions in much the same way that Agnew's work generalized Buck's theorem. We begin with the following lemma.

Lemma 2.1. *If x is a complex number sequence with a countably infinite set of (finite) limit points $D = \{\lambda_j\}_{j=1}^{\infty}$, then there is a subsequence y of x such that every λ_j in D is a statistical limit point of y.*

Proof. For every λ_j in D, there is a subsequence $\{x\}_{K(j)}$ of x such that $\{x\}_{K(j)}$ converges to λ_j. We construct y by choosing blocks of terms from the sets $\{x\}_{K(j)}$ in the following manner. The first block is chosen from $\{x\}_{K(1)}$. The second block is selected from $\{x\}_{K(1)}$ and the third block comes from $\{x\}_{K(2)}$. In general, the index of the set from which a block is selected follows the pattern $1,1,2,1,2,3,1,2,3,4,\dots$. The blocks are chosen so that the length of each block (after the initial one) is equal to the number of terms of y which precede it. We begin by selecting the first block $\{y_1\}$ (a single term) to be the first term of $\{x\}_{K(1)}$. Then we choose the second block $\{y_2\}$ to be the second term of $\{x\}_{K(1)}$. The third block consists of two terms, $\{y_3,y_4\}$, and is chosen from $\{x\}_{K(2)}$ so as to have the index of the x_j taken for y_3 larger than the indices used for y_1 and y_2, and to have y_4 be any term in $\{x\}_{K(2)}$ after x_j. For example, if the first two terms of $\{x\}_{K(1)}$ are x_{12} and x_{30}, we must choose for y_3 and y_4 terms x_j and x_k in $\{x\}_{K(2)}$ such that $j \ge 31$ and $k > j$. Otherwise, y would not be a subsequence of x because the original order of the chosen terms would not have been preserved. The fourth block of terms, $\{y_5,\dots,y_8\}$ is chosen from $\{x\}_{K(1)}$ so that the indices of the terms used are larger than those of any previously chosen x_j's. The fifth block, $\{y_9,\dots,y_{16}\}$ and the sixth block of terms, $\{y_{17},\dots,y_{32}\}$, are chosen from $\{x\}_{K(2)}$ and $\{x\}_{K(3)}$, respectively, with each term's index being larger than the indices of all of the x_j's which precede it. Having selected the nth block of terms from $\{x\}_{K(s)}$, that is, having constructed $\{y_1,\dots,y_q\}$ with $q = 2^{n-1}$, we choose the $(n+1)$st block of terms $\{y_{q+1},\dots,Y_{2q}\}$ from the set $\{x\}_{K(i)}$, where

$$i = \begin{cases} 1, & \text{if } n = \sum_{t=1}^{r} t \text{ for some } r \text{ in } \mathbb{N}, \\ s+1, & \text{otherwise.} \end{cases} \tag{2.1}$$

Here again, we must have each chosen term's index larger than the indices of all of the previously selected x_j's. This construction of $y = \{x_{n(k)}\}$ guarantees that $\{n(k)\}$ is a strictly increasing sequence of indices; so y is, indeed, a subsequence of x. We now must show that each λ_j in D is a statistical limit point of y. Consider a fixed but arbitrary j in $\mathbb{N}$. By selecting the blocks of terms as above, we have ensured that infinitely many blocks are chosen from $\{x\}_{K(j)}$. By concatenating these blocks in the order in which they appear in y, we get a subsequence w of y which converges to λ_j. This makes λ_j a limit point of y. To see that λ_j is a *statistical* limit point of y, we show that w is a nonthin subsequence of y. Notice that by the construction of y, whenever $n \ge 2$ the length of the nth block of terms $\{y_{q+1},\dots,y_{2q}\}$ is the same as the portion of y which precedes it, namely, $\{y_1,\dots,y_q\}$. So, for any block of terms chosen from $\{x\}_{K(j)}$ with ending term y_d, we have

$$\frac{1}{d}\left|\{k \le d : y_k \in \{x\}_{K(j)}\}\right| \ge \frac{1}{2}. \tag{2.2}$$

Therefore $\delta(\{k : y_k \in \{x\}_{K(j)}\}) \ne 0$, and so w is a nonthin subsequence of y. □

We can now state and prove the statistical convergence analogue to Agnew's original matrix result in [1].

Theorem 2.2. *If x is a complex number sequence and L_x is the set of (finite) limit points of x, then there is a subsequence y of x such that every λ in L_x is a statistical limit point of y.*

Proof. If L_x is a finite set, say $L_x = \{e_1,\dots,e_n\}$, let D be the countably infinite sequence $\{e_1,\dots,e_n,e_1,\dots,e_n,\dots\}$. If L_x is infinite, we use the separability of complex plane to find a countably infinite subset D of L_x such that the closure $\overline{D}$ of D is L_x. In either case, let y be the subsequence of x created in the proof of Lemma 2.1 using D. We only need to prove the result for the case when L_x is uncountable. Let λ_0 be a fixed but arbitrary element of L_x. Because $\overline{D} = L_x$, there is a sequence $\{\lambda_{n(i)}\}_{i=1}^{\infty}$ in D that converges to λ_0. By the construction of y, there are, whenever $i \geq 1$, infinitely many blocks of terms of y from $\{x\}_{K(n(i))}$. Also, since each $\{x\}_{K(n(i))}$ is a subsequence of x which converges to $\lambda_{n(i)}$, there are, for any positive number ε, infinitely many blocks of terms in y such that $|y_j - \lambda_{n(i)}| < \varepsilon/2$ for every y_j in each block. Moreover, since $\lim_i \lambda_{n(i)} = \lambda_0$, we can find, for any positive ε, an n_0 in $\mathbb{N}$ such that whenever $i \geq n_0, |\lambda_{n(i)} - \lambda_0| < \varepsilon/2$. Therefore, given a positive ε, we can find infinitely many blocks of terms of y such that $|y_j - \lambda_0| < \varepsilon$ for every y_j in each block. This then allows us to construct a nonthin subsequence w of y which converges to λ_0. We choose as our first block of w any block of y such that $|y_j - \lambda_0| < 1$ for all y_j in the block. We then choose the second block of w to be any block of y with $|y_j - \lambda_0| < 1/2$ for all y_j in the block and whose terms have indices larger than those of the first block. (We must concatenate the selected blocks in the order in which they appear in y, otherwise, w is not a subsequence of y.) Having chosen the first $n-1$ blocks of w, we choose the nth block of w to be any block of y, beyond those already chosen, with $|y_j - \lambda_0| < 1/n$ for all y_j in the block. Clearly, w is a subsequence of y with $\lim_q w_q = \lambda_0$. To see that w is a nonthin subsequence of y, recall from the construction of y in the proof of Lemma 2.1, that the length of any block of y is the same as the length of the portion of y which precedes it. Let J be the index set of w so that $w = \{y\}_J$. If we consider any block chosen in the construction of w (say, with ending term y_d), then

$$\frac{1}{d}\left|\{k \leq d : y_k \in \{y\}_J\}\right| \geq \frac{1}{2}. \tag{2.3}$$

Therefore, $\delta(J) \neq 0$ and w is a nonthin subsequence of y. □

We now show that a sequence x has a rearrangement z such that every limit point of x is a *statistical* limit point of z. The construction of z is similar in nature to the construction of y in the proof of Lemma 2.1, the major difference being that with a rearrangement, we must use every term of x exactly once. As was the case in the proof of Theorem 2.2, a lemma for the countable case is used in the proof of the general result.

Lemma 2.3. *If x is a complex number sequence with a countably infinite set of (finite) limit points $D = \{\lambda_j\}_{j=1}^{\infty}$, then there is a rearrangement z of x such that every λ_j in D is a statistical limit point of z.*

Proof. For every λ_j in D, there is a subsequence $\{x\}_{K(j)}$ of x such that $\{x\}_{K(j)}$ converges to λ_j. We construct z in a fashion similar to the construction of the subsequence y in the proof of Lemma 2.1. We choose blocks of terms from the sets $\{x\}_{K(i)}$

with the index i following this pattern: $1,1,2,1,2,3,1,2,3,4,\ldots$. Each block is chosen so its length is the same as that of the portion of z which precedes it. In between the ith and $(i+1)$st chosen blocks, we must use the terms of x that were skipped over while selecting the ith block. We begin by choosing the first block $\{z_1\}$ to be any term $x_{n(1)}$ of $\{x\}_{K(1)}$, where $n(1) \geq 2$. We then let $\{z_2,\ldots,z_{n(1)}\}$ be any permutation of the unused terms $\{x_1,\ldots,x_{n(1)-1}\}$. The second block of terms $\{z_{n(1)+1},\ldots,z_{2n(1)}\}$ is selected from $\{x\}_{K(1)}\setminus\{x_1,\ldots,x_{n(1)}\}$, where $x_{n(2)}$ has the largest index of the chosen terms. Then let $\{z_{2n(1)+1},\ldots,z_{n(2)}\}$ be any permutation of the terms x_j, where $n(1)+1 \leq j \leq n(2)-1$, that were not selected for the second block. We pick the third block of terms $\{z_{n(2)+1},\ldots,z_{2n(2)}\}$ from $\{x\}_{K(2)}\setminus\{x_1,\ldots,x_{n(2)}\}$ with $n(3)$ being the largest index of the selected x_j's. Let $\{z_{2n(2)+1},\ldots,z_{n(3)}\}$ be any permutation of the unused x_j's between $x_{n(2)+1}$ and $x_{n(3)-1}$. Once $\{z_1,\ldots,z_{n(k)}\}$ has been constructed, with the kth block of terms coming from $\{x\}_{K(s)}$, we select the $(k+1)$st block of terms $\{z_{n(k)+1},\ldots,z_{2n(k)}\}$ from $\{x\}_{K(i)}\setminus\{x_1,\ldots,x_{n(k)}\}$, where

$$i = \begin{cases} 1, & \text{if } k = \sum_{t=1}^{r} t \text{ for some } r \text{ in } \mathbb{N}, \\ s+1, & \text{otherwise.} \end{cases} \tag{2.4}$$

Let $n(k+1)$ be the largest index of the selected terms x_j, and let $\{z_{2n(k)+1},\ldots,z_{n(k+1)}\}$ be any permutation of the unused terms x_j between $x_{n(k)+1}$ and $x_{n(k+1)-1}$.

By the construction, $\{z_1,\ldots,z_{n(k)}\}$ is a permutation of $\{x_1,\ldots,x_{n(k)}\}$ whenever $k \geq 1$. Thus z is indeed a rearrangement of x. We now must show that each λ_j in D is a statistical limit point of z. Consider a fixed but arbitrary j in $\mathbb{N}$. In constructing z as above, we have ensured that infinitely many blocks are chosen from $\{x\}_{K(j)}$. By concatenating these blocks in the order in which they appear in z, we get a subsequence y of z which converges to λ_j. Notice that by the construction of z, whenever $i \geq 2$, the length of the ith block of terms is $n(i-1)$; which is precisely the length of the portion of z which precedes the ith block. So for any block of terms chosen from $\{x\}_{K(j)}$ with ending term z_d, we have

$$\frac{1}{d}\left|\{k \leq d : z_k \in \{x\}_{K(j)}\}\right| \geq \frac{1}{2}. \tag{2.5}$$

Therefore, $\delta(\{k : z_k \in \{x\}_{K(j)}\}) \neq 0$, so y is a nonthin subsequence of z converging to λ_j. Thus λ_j is a statistical limit point of z. □

The following theorem is a statistical convergence analogue to Fridy's rearrangement version of Agnew's matrix result. (See [9, Theorem 3].)

Theorem 2.4. *Let x be a complex number sequence and let L_x be the set of (finite) limit points of x, then there is a rearrangement z of x such that every λ in L_x is a statistical limit point of z.*

Proof. If L_x is a finite set, say, $L_x = \{e_1,\ldots,e_n\}$, let D be the countably infinite sequence $\{e_1,\ldots,e_n,e_1,\ldots,e_n,\ldots\}$. If L_x is infinite, we use the separability of the complex plane to find a countably infinite subset D of L_x such that the closure $\overline{D}$ of D is L_x. In either case, let z be the rearrangement of x created in the proof of Lemma 2.3 using D. We only need to prove the result for the case when L_x is uncountable. Let λ_0 be a fixed but arbitrary element of L_x. Using observations similar to those made in the proof of Theorem 2.2, we can find, for any positive number ε, infinitely many blocks of terms of z such that $|z_k-\lambda_0|<\varepsilon$ for every z_k in each block. This then allows us to create a subsequence y of z by concatenating blocks of z, in the order in which they appear, where $|z_k-\lambda_0|<1/n$ for all z_k in the nth block of y. Clearly, y converges to λ_0. To show that λ_0 is a statistical limit point of z, we need to show that y is a nonthin subsequence of z. We recall from the construction of z in Lemma 2.3, that the length of any block in z is equal to that of the portion of z which precedes it. Let J be the index set of y so that $y=\{z\}_J$. If we consider any block chosen in the construction of y (say, with ending term z_d), then

$$\frac{1}{d}\left|\{k\le d: z_k\in\{z\}_J\}\right|\ge\frac{1}{2}. \tag{2.6}$$

Therefore, $\delta(J)\ne 0$ and y is a nonthin subsequence of z. Thus λ_0 is a statistical limit point of z. □

To end this section, we give a result concerning stretching that is analogous to Theorems 2.2 and 2.4. However, before we can state and prove the result, we need to establish more notation. Let $x=\{x_n\}$ be a complex number sequence, let $\{x_{n(k)}\}$ be a subsequence of x, and let w be the stretching of x induced $\{m(p)\}_{p=0}^{\infty}$. Let

$$M=\bigcup_{k=1}^{\infty}\{m(n(k)-1),\dots,m(n(k))-1\}. \tag{2.7}$$

Definition 2.5. The subsequence $y[x_{n(k)}]=\{w\}_M$ of w is called the *subsequence corresponding to* $\{x_{n(k)}\}$ *in* w.

Notice that $w_q=x_{n(k)}$ whenever $m(n(k)-1)\le q<m(n(k))$. Thus if $\lim_k x_{n(k)}=L$, then $\lim_k y[x_{n(k)}]=L$.

Example 2.6. Consider the sequence $x=1,2,3,4,\dots$. The stretching w of x given by $w=1,1,1,2,3,3,4,4,4,4,\dots$ is induced by the sequence $m=1,4,5,7,11,\dots$; that is, $m(0)=1$, $m(1)=4$, $m(2)=5$, and so on. Let $x_{n(k)}=1,4,9,\dots$ be the subsequence of x consisting of the squares, i.e., $x_{n(k)}=k^2$. Using the notation from Definition 2.5 above, $y[x_{n(k)}]=\{1,1,1,4,4,4,4,\dots\}=\{w\}_M$, where

$$M=\{1,2,3\}\cup\{7,8,9,10\}\cup\cdots. \tag{2.8}$$

Lemma 2.7. *If x is a complex number sequence, then $\{2^p\}_{p=0}^{\infty}$ induces a stretching w of x in which for any subsequence $\{x_{n(k)}\}$ of x, $y[x_{n(k)}]$ is a nonthin subsequence of w.*

Proof. Let $\{x_{n(k)}\}$ be any subsequence of x and let

$$M=\bigcup_{k=1}^{\infty}\{2^{n(k)-1},\dots,2^{n(k)}-1\}. \tag{2.9}$$

Whenever $k \geq 1$,

$$\begin{aligned}\frac{1}{2^{n(k)}-1}\left|\{q \leq 2^{n(k)}-1 : q \in M\}\right| &\geq \frac{1}{2^{n(k)}-1}\left|\{q \leq 2^{n(k)}-1 : w_q = x_{n(k)}\}\right| \\ &\geq \frac{2^{n(k)}-1-2^{n(k)-1}+1}{2^{n(k)}-1} \geq \frac{2^{n(k)-1}}{2^{n(k)}-1} > \frac{1}{2}.\end{aligned} \tag{2.10}$$

Thus $\delta(M) \neq 0$ and hence $y[x_{n(k)}] = \{w\}_M$ is a nonthin subsequence of w. □

Here then is the statistical convergence analogue to Dawson's stretching version of Agnew's theorem. (See [6, Theorem 3].)

Theorem 2.8. *If x is a complex number sequence, then $\{2^p\}_{p=0}^{\infty}$ induces a stretching w of x in which every (finite) limit point of x is a statistical limit point.*

Proof. Let λ be a (finite) limit point of x, and let w be the stretching of x induced by $\{2^p\}_{p=0}^{\infty}$. Then there is a subsequence $\{x_{n(k)}\}$ of x that converges to λ. By Lemma 2.7, $y[x_{n(k)}]$ is a nonthin subsequence of w which converges to λ. Thus λ is a statistical limit point of w. □

It should be noted that $\{2^p\}$ is independent of x, that is, given *any* sequence x, the stretching w induced by $\{2^p\}$ has the desired statistical limit points. However, the constructions of a subsequence y and rearrangement z with the appropriate statistical limit points depended on the given sequence x.

3. A-statistical limit point theorems.

In [12, Theorem 2.3], it is shown that for a nonnegative regular matrix A, a sequence x is A-statistically convergent to λ if and only if there is an infinite index set K with $\delta_A(K) = 1$ such that $\{x\}_K$ converges to λ. This then implies that if x is A-statistically convergent to λ, then λ is the only A-statistical limit point of x. Now, we generalize the constructions given in [17, Theorem 2, 4, 6] to give A-statistical convergence analogues to Agnew's theorem [1] and its analogues for rearrangements [9, Theorem 3] and stretchings [6, Theorem 3]. We begin by proving the following lemma.

Lemma 3.1. *Let x be a complex number sequence with a countably infinite set of (finite) limit points $D = \{\lambda_j\}_{j=1}^{\infty}$. Given a nonnegative regular matrix A, there exists a subsequence y of x such that every λ_j in D is an A-statistical limit point of y.*

Proof. Throughout the proof, A is a fixed but arbitrary nonnegative regular matrix. For every λ_j in D there is a subsequence $\{x\}_{M(j)}$ of x such that $\{x\}_{M(j)}$ converges to λ_j. Using this countable collection of sets, we construct the subsequence y in a manner very similar to the construction found in the proof of Lemma 2.1. Blocks of terms for y are chosen from the sets $\{x\}_{M(j)}$ with the index j following this pattern: $1,1,2,1,2,3,1,2,3,4,\ldots$. Using the regularity of A, we select two strictly increasing sequence of indices $\{n(i)\}_{i=0}^{\infty}$ and $\{k(i)\}_{i=0}^{\infty}$ such that $k(0) = 0$ and

$$\sum_{k=1+k(i)}^{k(i+1)} a_{n(i+1),k} > \frac{1}{2} \quad \text{forall } i = 0,1,2,\ldots. \tag{3.1}$$

With these indices, we construct the subsequence y as follows. Let $\{y_1,\ldots,y_{k(1)}\}$ be the first $k(1)$ terms of $\{x\}_{M(1)}$ taken in the order in which they appear in x.

Let $\{y_{1+k(1)},\dots,y_{k(2)}\}$ be the next $k(2)-k(1)$ terms of $\{x\}_{M(1)}$, taken in order. Select $\{y_{1+k(2)},\dots,y_{k(3)}\}$ as the first $k(3)-k(2)$ terms of $\{x\}_{M(2)}$, taken in order, whose indices are larger than that of the x_p chosen as $y_{k(2)}$. The next three blocks $\{y_{1+k(3)},\dots,y_{k(4)}\}$, $\{y_{1+k(4)},\dots,y_{k(5)}\}$ and $\{y_{1+k(5)},\dots,y_{k(6)}\}$ are chosen from $\{x\}_{M(1)}$, $\{x\}_{M(2)}$ and $\{x\}_{M(3)}$, respectively, with each term's index being larger than the indices of all of the x_j's which precede it.

Having selected the mth block of terms $\{y_{1+k(m-1)},\dots,y_{k(m)}\}$ from $\{x\}_{M(s)}$, we then choose the $(m+1)$st block of terms $\{y_{1+k(m)},\dots,y_{k(m+1)}\}$ from $\{x\}_{M(i)}$, where

$$i=\begin{cases}1, & \text{if } k=\sum_{t=1}^{r} t \text{ for some } r \text{ in } \mathbb{N},\\ s+1, & \text{otherwise.}\end{cases} \tag{3.2}$$

Here again, we must have the index of each chosen term larger than the indices of all of the previously selected x_j's so that y is a subsequence of x. We must show that each λ_j in D is an A-statistical limit point of y. Consider a fixed but arbitrary j in $\mathbb{N}$. By selecting the blocks of terms as above, we have ensured that infinitely many blocks are chosen from $\{x\}_{M(j)}$. By concatenating these blocks in the order in which they appear in y, we get a subsequence z of y which converges to λ_j. Thus λ_j is a limit point of y. To see that λ_j is an A-*statistical* limit point of y, we show that z is an A-nonthin subsequence of y. Let $K=\{k : y_k\in\{x\}_{M(j)}\}$. Notice that by the selection of $\{n(i)\}$ and $\{k(i)\}$ and by the construction of y, for indices $n(d)$, where $y_{k(d)}$ is the last term of a block chosen from $\{x\}_{M(j)}$, we have

$$\sum_{k\in K} a_{n(d),k}\ge\sum_{k=1+k(d-1)}^{k(d)} a_{n(d),k}>\frac{1}{2}. \tag{3.3}$$

Thus for infinitely many indices n we have $\sum_{k\in K} a_{n,k}>1/2$. Therefore $\delta_A(K)\neq 0$, and so z is an A-nonthin subsequence of y. □

We now state and prove the A-statistical convergence analogue to Agnew's theorem [1].

Theorem 3.2. *Let x be a complex number sequence and let L_x be the set of (finite) limit points of x. Given a nonnegative regular matrix A, there exists a subsequence y of x such that every λ in L_x is an A-statistical limit point of y.*

Proof. If L_x is a finite set, say $L_x=\{e_1,\dots,e_n\}$, let D be the countably infinite sequence $\{e_1,\dots,e_n,e_1,\dots,e_n,\dots\}$. If L_x is infinite, we use the separability of the complex plane to find a countably infinite subset D of L_x such that the closure $\overline{D}$ of D is L_x. In either case, let y be the subsequence of x created in the proof of Lemma 3.1 using D. We need only prove the result for the case when L_x is uncountable. Consider an arbitrary but fixed λ_0 in L_x. Using observations similar to those made in the proof of Theorem 2.2, we can find, for any positive number ε, infinitely many blocks of terms of y such that $|y_k-\lambda_0|<\varepsilon$ for every y_k in each block. This then allows us to create a subsequence z of y by concatenating blocks of y, in the order in which they appear, where $|y_k-\lambda_0|<1/n$ for all y_k in the nth block of z. Clearly, z converges to λ_0.To show that λ_0 is an A-statistical limit point of y, we need to show that z is an

A-nonthin subsequence of y. Let J be the index set of z so that $z = \{y\}_J$. For indices $n(d)$, where $y_{k(d)}$ is the last term of a block chosen for z, we have

$$\sum_{k\in J} a_{n(d),k} \geq \sum_{k=1+k(d-1)}^{k(d)} a_{n(d),k} > \frac{1}{2}. \tag{3.4}$$

Thus for infinitely many indices n, we have $\sum_{k\in J} a_{n,k} > 1/2$. Therefore

$$\delta_A(J) = \delta_A(\{k : y_k \in \{y\}_J\}) \neq 0, \tag{3.5}$$

and so z is an A-nonthin subsequence of y. □

Our next goal is to show that, given a fixed nonnegative regular matrix A, a sequence x has a rearrangement z such that every limit point of x is an A-*statistical* limit point of z. Here again, the construction of the rearrangement z is similar in nature to the construction of the subsequence y above, the major difference being that with a rearrangement, we must use every term of x exactly once. We first prove the following lemma which is used in the proof of the general result.

Lemma 3.3. *Let x be a complex number sequence with a countably infinite set of (finite) limit points $D = \{\lambda_j\}_{j=1}^{\infty}$. Given a nonnegative regular matrix A, there exists a rearrangement z of x such that every λ_j in D is an A-statistical limit point of z.*

Proof. Throughout the proof, A is a fixed but arbitrary nonnegative regular matrix. For every λ_j in D there is a subsequence $\{x\}_{M(j)}$ of x such that $\{x\}_{M(j)}$ converges to λ_j. We construct z by choosing blocks of terms from the sets $\{x\}_{M(j)}$ with the index j following this pattern: $1,1,2,1,2,3,1,2,3,4,\ldots$. In between the ith and $(i+1)$st chosen blocks, we must use the terms of x that were skipped over while selecting the ith block. Let $J(1) = 1$. Choose $N(1)$ and then $K(1)$ such that $K(1) > J(1)$ and

$$\sum_{k=J(1)}^{K(1)} a_{N(1),k} > \frac{1}{2}. \tag{3.6}$$

Select $\{z_{J(1)},\ldots,z_{K(1)}\}$ as any permutation of the first $K(1)$ terms of $\{x\}_{M(1)}$. Let m be the largest index of the terms x_k chosen from $\{x\}_{M(1)}$ for the block and set $J(2) = m+1$. Notice that $J(2) > K(2)$. Next, let $\{z_{K(1)+1},\ldots,z_{J(2)-1}\}$ be any permutation of the unused terms x_j, where $J(1) \leq j \leq J(2)-1$. Select $N(2)$ and then $K(2)$ such that $N(2) > N(1)$, $K(2) > J(2)$ and

$$\sum_{k=J(2)}^{K(2)} a_{N(2),k} > \frac{1}{2}. \tag{3.7}$$

Choose $\{z_{J(2)},\ldots,z_{K(2)}\}$ as any permutation of the first $K(2)-J(2)+1$ terms of $\{x_{M(1)}\}\backslash\{x_1,\ldots,x_{J(2)-1}\}$. Let m be the largest index of the terms chosen from $\{x\}_{M(1)}$ thus far and set $J(3) = m+1$. Select $\{z_{K(2)+1},\ldots,z_{J(3)-1}\}$ as any permutation of the unused terms x_j, where $J(2) \leq j \leq J(3)-1$. Choose $N(3)$ and then $K(3)$ such that $N(3) > N(2)$, $K(3) > J(3)$ and

$$\sum_{k=J(3)}^{K(3)} a_{N(3),k} > \frac{1}{2}. \tag{3.8}$$

Let $\{z_{J(3)},\dots,z_{K(3)}\}$ be a permutation of the first $K(3)-J(3)+1$ terms of $\{x_{M(2)}\}\setminus\{x_1,\dots,x_{J(3)-1}\}$. Let m be the largest index of the terms x_k chosen from $\{x\}_{M(2)}$ for the block and set $J(4)=m+1$. Choose $\{z_{K(3)+1},\dots,z_{J(4)-1}\}$ to be any permutation of the unused terms x_j, where $J(3)\le j\le J(4)-1$.

Having constructed $\{z_1,\dots,z_{J(n)-1}\}$, with the $(n-1)$st block $\{z_{J(n-1)},\dots,z_{K(n-1)}\}$ coming from $\{x\}_{M(s)}$, and thus having chosen indices $\{N(p)\}_{p=1}^{n-1}$, $\{J(p)\}_{p=1}^{n}$ and $\{K(p)\}_{p=1}^{n-1}$ such that

$$\begin{gathered} N(1)<N(2)<\cdots<N(n-1), \\ J(1)<K(1)<J(2)<\cdots<J(n-1)<K(n-1)<J(n), \end{gathered} \tag{3.9}$$

choose $N(n)$ and then $K(n)$ such that $N(n)>N(n-1)$, $K(n)>J(n)$ and

$$\sum_{k=J(n)}^{K(n)} a_{N(n),k}>\frac{1}{2}. \tag{3.10}$$

The nth block of terms $\{z_{J(n)},\dots,z_{K(n)}\}$ is chosen as any permutation of the first $K(n)-J(n)+1$ terms of $\{x_{M(i)}\}\setminus\{x_1,\dots,x_{J(n)-1}\}$, where

$$i=\begin{cases} 1, & \text{if } k=\displaystyle\sum_{t=1}^{r} t \text{ for some } r \text{ in } \mathbb{N}, \\ s+1, & \text{otherwise.} \end{cases} \tag{3.11}$$

Let m be the largest index of the terms x_k chosen from $\{x_{M(i)}\}$ for the block and set $J(n+1)=m+1$. Select the block of terms $\{z_{K(n)+1},\dots,z_{J(n+1)-1}\}$ as any permutation of the unused terms x_j, where $J(n)\le j\le J(n+1)-1$.

By the construction, $\{z_1,\dots,z_{J(n)-1}\}$ is a permutation of $\{x_1,\dots,x_{J(n)-1}\}$ whenever $n\ge 1$. Thus z is indeed a rearrangement of x. We now must show that each λ_j in D is an A-statistical limit point of z. Consider a fixed but arbitrary j in $\mathbb{N}$. In constructing z as above, we have ensured that infinitely many blocks are chosen from $\{x\}_{M(j)}$. By concatenating these blocks in the order in which they appear in z, we get a subsequence y of z which converges to λ_j. Thus λ_j is a limit point of z. Let $K=\{k : z_k\in\{x\}_{M(j)}\}$. Notice that by the selection of $\{N(p)\}_{p=1}^{\infty}$, $\{J(p)\}_{p=1}^{\infty}$ and $\{K(p)\}_{p=1}^{\infty}$ and by the construction of z, that for indices $N(d)$, $J(d)$ and $K(d)$, where $\{z_{J(d)},\dots,z_{K(d)}\}$ is a block of z chosen from $\{x\}_{M(j)}$, we have

$$\sum_{k\in K} a_{N(d),k}\ge\sum_{k=J(d)}^{K(d)} a_{N(d),k}>\frac{1}{2}. \tag{3.12}$$

Thus for infinitely many indices n, we have $\sum_{k\in K} a_{n,k}>1/2$. Therefore $\delta_A(K)\ne 0$, and so y is an A-nonthin subsequence of z and λ_j is an A-*statistical* limit point of z. □

We now can state and prove the A-statistical convergence analogue to Fridy's rearrangement version of Agnew's theorem. (See [9, Theorem 3].)

Theorem 3.4. *Let x be a complex number sequence and let L_x be the set of (finite) limit points of x. Given a nonnegative regular matrix A, there exists a rearrangement z of x such that every λ in L_x is an A-statistical limit point of z.*

Proof. If L_x is a finite set, say $L_x = \{e_1,\dots,e_n\}$, let D be the countably infinite sequence $\{e_1,\dots,e_n,e_1,\dots,e_n,\dots\}$. If L_x is infinite, we use the separability of the complex plane to find a countably infinite subset D of L_x such that the closure $\overline{D}$ of D is L_x. In either case, let z be the rearrangement of x created in the proof of Lemma 3.3 using D. We only need to prove the result for the case when L_x is uncountable. Consider an arbitrary but fixed λ_0 in L_x. Using observations similar to those made in the proof of Theorem 2.2, we can find, for any positive number ε, infinitely many blocks of terms of z such that $|z_k - \lambda_0| < \varepsilon$ for every z_k in each block. This then allows us to create a subsequence y of z by concatenating blocks of z, in the order in which they appear, where $|z_k - \lambda_0| < 1/n$ for all z_k in the nth block of y. Clearly, y converges to λ_0. To show that λ_0 is an A-statistical limit point of z, we need to show that y is an A-nonthin subsequence of z. Let K be the index set of y so that $y = \{z\}_K$. If we consider any block chosen in the construction of y, say $\{z_{J(d)},\dots,z_{K(d)}\}$, then

$$\sum_{k\in K} a_{N(d),k} \geq \sum_{k=J(d)}^{K(d)} a_{N(d),k} > \frac{1}{2}. \tag{3.13}$$

Thus for infinitely many indices n, we have $\sum_{k\in K} a_{n,k} > 1/2$. Therefore

$$\delta_A(K) = \delta_A\left(\{k : z_k \in \{z\}_K\}\right) \neq 0, \tag{3.14}$$

and so y is an A-nonthin subsequence of z. □

To finish the section, we give an A-statistical convergence analogue to Dawson's stretching version of Agnew's theorem. (See [6, Theorem 3].)

Theorem 3.5. *Let x be a complex number sequence and let L_x be the set of (finite) limit points of x. Given a nonnegative regular matrix A, there exists a stretching w of x such that every λ in L_x is an A-statistical limit point of w.*

Proof. Using the regularity of A, we choose strictly increasing sequences of indices $\{m(p)\}_{p=0}^{\infty}$ and $\{N(p)\}_{p=1}^{\infty}$ such that $m(0) = 1$ and

$$\sum_{m(p-1)}^{m(p)-1} a_{N(p),k} > \frac{1}{2}, \tag{3.15}$$

whenever $p \geq 1$. Let w be the stretching induced by $\{m(p)\}_{p=0}^{\infty}$. Given a fixed but arbitrary element λ_0 of L_x, there is subsequence $\{x_{n(j)}\}_{j=1}^{\infty}$ of x that converges to λ_0. Let

$$M = \bigcup_{j=1}^{\infty} \{m(n(j)-1),\dots,m(n(j))-1\}. \tag{3.16}$$

(See Example 2.6 as an example of this notation.) The subsequence $\{w\}_M$ of w converges to λ_0 because $w_q = x_{n(j)}$, whenever $m(n(j)-1) \leq q < m(n(j))$, so λ_0 is a limit point of w. To show that λ_0 is an A-*statistical* limit point of w, we need to show that $\{w\}_M$ is A-nonthin in w. Notice that whenever $j \geq 1$,

$$\sum_{k\in M} a_{N(n(j)),k} \geq \sum_{k=m(n(j)-1)}^{m(n(j))-1} a_{N(n(j)),k} > \frac{1}{2}. \tag{3.17}$$

Hence $\delta_A(M) \neq 0$ and thus $\{w\}_M$ is an A-nonthin subsequence of w. □

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