Internat. J. Math. & Math. Sci.
Vol. 23, No. 4 (2000) 279–283
S0161171200000624

SKEW GROUP RINGS WHICH ARE GALOIS

GEORGE SZETO and LIANYONG XUE

(Received 20 March 1998)

ABSTRACT. Let $S*G$ be a skew group ring of a finite group G over a ring S. It is shown that if $S*G$ is an G'-Galois extension of $(S*G)^{G'}$, where G' is the inner automorphism group of $S*G$ induced by the elements in G, then S is a G-Galois extension of S^G. A necessary and sufficient condition is also given for the commutator subring of $(S*G)^{G'}$ in $S*G$ to be a Galois extension, where $(S*G)^{G'}$ is the subring of the elements fixed under each element in G'.

Keywords and phrases. Skew group rings, Azumaya algebras, Galois extensions, H-separable extensions.

2000 Mathematics Subject Classification. Primary 16S35; Secondary 16W20.

1. Introduction. Let S be a ring with 1, C the center of S, G a finite automorphism group of S of order n invertible in S, S^G the subring of the elements fixed under each element in G, $S*G$ a skew group ring of group G over S, and G' the inner automorphism group of $S*G$ induced by the elements in G, that is, $g'(x)=gxg^{-1}$ for each g in G and x in $S*G$, so the restriction of G' to S is G. In [3, 2], a G-Galois extension S of S^G which is an Azumaya C^G-algebra is characterized in terms of the Azumaya C^G-algebra $S*G$ and the H-separable extension $S*G$ of S, respectively, and the properties of the commutator subring of S in $S*G$ are given in [1]. It is clear that S is a G-Galois extension of S^G implies that $S*G$ is a G'-Galois extension of $(S*G)^{G'}$ with the same Galois system as S. In the present paper, we prove the converse theorem: if $S*G$ is a G'-Galois extension of $(S*G)^{G'}$, then S is a G-Galois extension of S^G. Moreover, for a G'-Galois extension $S*G$ of $(S*G)^{G'}$ which is a projective separable C^G-algebra, S can be shown to be a G-Galois extension of S^G which is also a projective separable C^G-algebra. Then a sufficient condition on $(S*G)^{G'}$ is given for S to be a G-Galois extension of S^G which is an Azumaya C^G-algebra, and an equivalent condition on S^G is obtained for the commutator subring of $(S*G)^{G'}$ in $S*G$ to be a G-Galois extension.

2. Preliminaries. Throughout, we keep the notation as given in the introduction. Let B be a subring of a ring A with 1. Following [3, 2], A is called a separable extension of B if there exist $\{a_i, b_i$ in A, $i=1,2,\dots,m$ for some integer $m\}$ such that $\sum a_ib_i=1$, and $\sum sa_i\otimes b_i=\sum a_i\otimes b_is$ for all s in A, where $\otimes$ is over B. An Azumaya algebra is a separable extension of its center. A ring A is called an H-separable extension of B if $A\otimes_B A$ is isomorphic to a direct summand of a finite direct sum of A as an A-bimodule. It is known that an Azumaya algebra is an H-separable extension and an H-separable extension is a separable extension. Let S be given as in Section 1.

Then it is called a G-Galois extension of S^G if there exist elements $\{c_i, d_i$ in S, $i = 1, 2, \ldots, k$ for some integer $k\}$ such that $\sum c_i g_j(d_i) = \delta_{1,j}$, where $G = \{g_1, g_2, \ldots, g_n\}$ with identity g_1, for each $g_j \in G$. Such a set $\{c_i, d_i\}$ is called a G-Galois system for S.

3. Galois skew group rings. In this section, we show that a G'-Galois extension skew group ring $S*G$ implies a G-Galois extension S. More results are obtained for S^G when $(S*G)^{G'}$ is a projective separable C^G-algebra, and an H-separable S^G-extension, respectively.

Theorem 3.1. *If $S*G$ is a G'-Galois extension of $(S*G)^{G'}$, then S is a G-Galois extension of S^G.*

Proof. Let $\{u_i, v_i \mid i = 1, 2, \ldots, m\}$ be a G'-Galois system of $S*G$ over $(S*G)^{G'}$, that is, u_i and v_i are elements of $S*G$ satisfying $\sum_{i=1}^m u_i g'(v_i) = \sum_{i=1}^m u_i g v_i g^{-1} = \delta_{1,g}$. Let $w_i = \sum_{h\in G} h v_i$, $i = 1, 2, \ldots, m$. Then $g w_i = \sum_{h\in G} g h v_i = w_i$. Since $\{h \mid h \in G\}$ is a basis of $S*G$ over S, we have $u_i = \sum_{h\in G} s_h^{(u_i)} h$ and $w_i = \sum_{h\in G} s_h^{(w_i)} h$, $i = 1, 2, \ldots, m$, for some $s_h^{(u_i)}$, $s_h^{(w_i)}$ in S. Let $x_i = \sum_{h\in G} s_h^{(u_i)}$ and $y_i = s_1^{(w_i)}$, $i = 1, 2, \ldots, m$. We prove that $\{x_i, y_i \mid i = 1, 2, \ldots, m\}$ is a G-Galois system for S over S^G. First, we prove that

(1) $g(s_h^{(w_i)}) = s_{gh}^{(w_i)}$ for all $i = 1, 2, \ldots, m$ and all $g, h \in G$,

(2) $\sum_{i=1}^m u_i w_i = 1$.

For (1), since $w_i = g w_i$, we have

$$\sum_{k\in G} s_k^{(w_i)} k = \sum_{h\in G} s_h^{(w_i)} h = g \sum_{h\in G} s_h^{(w_i)} h = \sum_{h\in G} g s_h^{(w_i)} h = \sum_{h\in G} g\left(s_h^{(w_i)}\right) gh. \tag{3.1}$$

Since $\{k \mid k \in G\}$ is a basis of $S*G$ over S, $g(s_h^{(w_i)}) = s_{gh}^{(w_i)}$.

For (2), since $\{u_i, v_i \mid i = 1, 2, \ldots, m\}$ is a G'-Galois system for $S*G$ over $(S*G)^{G'}$, $\sum_{i=1}^m u_i h'(v_i) \sum_{i=1}^m u_i h v_i h^{-1} = \delta_{1,h}$. Therefore,

$$1 = \sum_{h\in G} \delta_{1,h} h = \sum_{h\in G} \left(\sum_{i=1}^m u_i h v_i h^{-1} \right) h = \sum_{h\in G} \sum_{i=1}^m u_i h v_i = \sum_{i=1}^m u_i \sum_{h\in G} h v_i = \sum_{i=1}^m u_i w_i. \tag{3.2}$$

Next, we prove that $\{x_i, y_i \mid i = 1, 2, \ldots, m\}$ is a G-Galois system for S over S^G. By using (1) and (2), we get

$$\begin{aligned}
1 &= \sum_{i=1}^m u_i w_i = \sum_{i=1}^m \left(\sum_{h\in G} s_h^{(u_i)} h \right) \left(\sum_{k\in G} s_k^{(w_i)} k \right) \\
&= \sum_{i=1}^m \sum_{h\in G} \sum_{k\in G} s_h^{(u_i)} h s_k^{(w_i)} k = \sum_{i=1}^m \sum_{h\in G} \sum_{k\in G} s_h^{(u_i)} h\left(s_k^{(w_i)}\right) hk \\
&= \sum_{i=1}^m \sum_{g\in G} \sum_{hk=g} s_h^{(u_i)} h\left(s_k^{(w_i)}\right) hk = \sum_{i=1}^m \sum_{g\in G} \sum_{hk=g} s_h^{(u_i)} s_{hk}^{(w_i)} hk \quad \text{by (1)} \\
&= \sum_{i=1}^m \sum_{g\in G} \sum_{h\in G} s_h^{(u_i)} s_{hh^{-1}g}^{(w_i)} hh^{-1} g \quad (\text{since } hk = g,\ k = h^{-1} g) \\
&= \sum_{i=1}^m \sum_{g\in G} \sum_{h\in G} s_h^{(u_i)} s_g^{(w_i)} g = \sum_{g\in G} \left(\sum_{i=1}^m \sum_{h\in G} s_h^{(u_i)} s_g^{(w_i)} \right) g.
\end{aligned} \tag{3.3}$$

Hence, $\sum_{i=1}^{m}\sum_{h\in G} s_h^{(u_i)} s_g^{(w_i)} = \delta_{1,g}$. But $x_i = \sum_{h\in G} s_h^{(u_i)}$, $y_i = s_1^{(w_i)}$, and $g(s_1^{(w_i)}) = s_g^{(w_i)}$ by (1). So,

$$\sum_{i=1}^{m} x_i g(y_i) = \sum_{i=1}^{m}\sum_{h\in G} s_h^{(u_i)} g\left(s_1^{(w_i)}\right) = \sum_{i=1}^{m}\sum_{h\in G} s_h^{(u_i)} s_g^{(w_i)} = \delta_{1,g}. \tag{3.4}$$

□

We show more properties of the G-Galois extension S of S^G when $S*G$ is a G'-Galois extension of $(S*G)^{G'}$ which possesses a property.

Theorem 3.2. *If $S*G$ is a G'-Galois extension of $(S*G)^{G'}$ which is a projective separable C^G-algebra, then S is a G-Galois extension of S^G which is also a projective separable C^G-algebra.*

Proof. Since $S*G$ is a G'-Galois extension of $(S*G)^{G'}$, S is a G-Galois extension of S^G by Theorem 3.1. Again, since $S*G$ is a G'-Galois extension of $(S*G)^{G'}$, it is a separable extension [5]. Also, $(S*G)^{G'}$ is a separable C^G-algebra, so $S*G$ is a separable C^G-algebra by the transitivity of separable extensions. Next, we claim that S is also a separable C^G-algebra. In fact, since n is a unit in S, the trace map: $(1/n)(\mathrm{tr}_G(\ \ ))$: $S \to S^G \to 0$ is a splitting homomorphism of the imbedding homomorphism of S^G into S as a two sided S^G-module. Hence, S^G is a direct summand of S. Since S is a direct summand of $S*G$ as an S^G-bimodule, S^G is so of $S*G$ as an S^G-module. Moreover, S is a finitely generated and projective S^G-module (for S is a G-Galois extension of S^G), so $S*G$ is a finitely generated and projective S^G-module by the transitivity of the finitely generated and projective modules. This implies that S^G is a projective separable C^G-algebra by [5, proof of Lem. 2, p. 120]. □

Theorem 3.3. *If*

(i) *$S*G$ is a G'-Galois extension of $(S*G)^{G'}$*

(ii) *$(S*G)^{G'}$ is an H-separable extension of S^G which is a separable C^G-algebra, then S is a G-Galois extension of S^G which is an Azumaya C^G-algebra.*

Proof. Since $S*G$ is a G'-Galois extension of $(S*G)^{G'}$ with an inner Galois group G', $S*G$ is an H-separable extension of $(S*G)^{G'}$ [7, Prop. 4]. By hypothesis, $(S*G)^{G'}$ is an H-separable extension of S^G, so $S*G$ is an H-separable extension of S^G by the transitivity of H-separable extensions. Noting that n is a unit of S, we have S^G is an S^G-direct summand of S. But S is a direct summand of $S*G$ as an S^G-module, so S^G is a direct summand of $S*G$ as an S^G-module. Thus, $V_{S*G}(V_{S*G}(S^G)) = S^G$ [6, Prop. 1.2]. This implies that the center of $S*G$ is contained in S^G, and so the center of $S*G$ is C^G. Therefore, $S*G$ is an Azumaya C^G-algebra. Thus, S^G is an Azumaya C^G-algebra. Consequently, S is a G-Galois extension of S^G which is an Azumaya C^G-algebra [2, Thm. 3.1]. □

4. Galois commutator subrings. In [7], the class of G-Galois and H-separable extension was studied. Let A be a G-Galois and H-separable extension of A^G and let $V_A(A^G)$ be the commutator subring of A^G in A. Then, $V_A(A^G)$ is a central (G/I)-Galois algebra if and only if $A^I = A^G(V_A(A^G))$, where $I = \{g \in G \mid g(d) = d \text{ for all } d \in V_A(A^G)\}$ [7, Thm. 6.3]. Applying such an equivalence condition to a G'-Galois extension $S*G$, we characterize a Galois commutator subring $V_{S*G}((S*G)^{G'})$ in terms of elements in S^G.

In the following, we denote the center of G by P and the center $S*G$ by Z. By a direct computation, we have the following.

Lemma 4.1. (1) *Let* $I=\{g_i\in G\mid g_i'(d)=d$ *for each* $d\in ZG\}$. *Then* $I=P$.

(2) *Let* x *be an element in* $(S*G)^{G'}$. *Then* $x=\sum_{i=1}^n s_ig_i$ *such that* $g_j(s_i)=s_k$ *whenever* $g_jg_ig_j^{-1}=g_j'(g_i)=g_k\in G$.

Lemma 4.2. *Assume that* $S*G$ *is a* G'*-Galois extension of* $(S*G)^{G'}$ *and an Azumaya* Z*-algebra. Then* $V_{S*G}((S*G)^{G'})$ *is a central* (G'/P')*-Galois algebra if and only if* $S^PG=(S*G)^{G'}G$.

Proof. Since n is a unit in Z and $S*G$ is an Azumaya Z-algebra, $V_{S*G}((S*G)^{G'})=V_{S*G}(V_{S*G}(ZG))=ZG$ by the commutator theorem for Azumaya algebras [4, Thm. 4.3] (for ZG is a separable Z-subalgebra). Moreover, since $S*G$ is a G'-Galois extension of $(S*G)^{G'}$ with an inner Galois group G', it is an H-separable extension of $(S*G)^{G'}$ [7, Prop. 4]. Hence, $V_{S*G}((S*G)^{G'})(=ZG)$ is a central (G'/P')-Galois algebra if and only if $(S*G)^{P'}=(S*G)^{G'}ZG$ by [7, Lem. 4.1(1) and Thm. 6.3]. Clearly, $Z\subset(S*G)^{G'}$, and so $(S*G)^{G'}ZG=(S*G)^{G'}G$. Noting that P is the center of G, we have $(S*G)^{P'}=S^PG$. Thus, the lemma holds. □

Theorem 4.4. *Assume that* $S*G$ *is a* G'*-Galois extension of* $(S*G)^{G'}$ *and an Azumaya* Z*-algebra. Then* ZG *is a central* (G'/P')*-Galois algebra if and only if, for every* $s\in S^P$, *there exists an* $n\times n$ *matrix* $[s_{k,h}]_{k,h\in G}$ *for some* $s_{k,h}$ *in* S *such that*

(1) $\sum_{h\in G}s_{gh^{-1},h}=\delta_{1,g}s$ *(therefore,* $s=\sum_{h\in G}s_{h^{-1},h}$*), and*

(2) $g(s_{k,h})=s_{gkg^{-1},h}$ *for every* $g\in G$.

Proof. ($\Longrightarrow$) Assume that ZG is a central (G'/P')-Galois algebra. Then by Lemma 4.2, $S^PG=(S*G)^{G'}G$. Therefore, for every $s\in S^P$, $s=s1\in S^PG=(S*G)^{G'}G$. Hence, there exists $\sum_{k\in G}s_{k,h}k\in(S*G)^{G'}$ for each $h\in G$ such that

$$s=s1=\sum_{h\in G}\left(\sum_{k\in G}s_{k,h}k\right)h=\sum_{g\in G}\left(\sum_{kh=g}s_{k,h}\right)g=\sum_{g\in G}\left(\sum_{h\in G}s_{gh^{-1},h}\right)g. \tag{4.1}$$

Since $\{g\mid g\in G\}$ is a basis of $S*G$ over S, we have $\sum_{h\in G}s_{gh^{-1},h}=\delta_{1,g}s$ and, therefore, $\sum_{h\in G}s_{h^{-1},h}=s$. Furthermore, for each $h\in G$, $\sum_{k\in G}s_{k,h}k\in(S*G)^{G'}$, i.e., $\sum_{k\in G}s_{k,h}k=g\sum_{k\in G}s_{k,h}kg^{-1}=\sum_{k\in G}g(s_{k,h})gkg^{-1}$ for every $g\in G$. Therefore, $g(s_{k,h})=s_{gkg^{-1},h}$ for every $g\in G$ since $\{k\mid k\in G\}$ is a basis of $S*G$ over S.

($\Longleftarrow$) Assume that, for every $s\in S^P$, there exists an $n\times n$ matrix $[s_{k,h}]_{k,h\in G}$ such that $\sum_{h\in G}s_{gh^{-1},h}=\delta_{1,g}s$ and $g(s_{k,h})=s_{gkg^{-1},h}$ for every $g\in G$. Then

$$g\left(\sum_{k\in G}s_{k,h}k\right)g^{-1}=\sum_{k\in G}g(s_{k,h})gkg^{-1}=\sum_{k\in G}s_{gkg^{-1},h}gkg^{-1}=\sum_{k\in G}s_{k,h}k, \tag{4.2}$$

that is $\sum_{k\in G}s_{k,h}k\in(S*G)^{G'}$ for every $h\in G$. Therefore,

$$\begin{aligned}s&=\sum_{g\in G}\delta_{1,g}sg=\sum_{g\in G}\left(\sum_{h\in G}s_{gh^{-1},h}\right)g=\sum_{g\in G}\left(\sum_{kh=g}s_{k,h}\right)g\\&=\sum_{h\in G}\left(\sum_{k\in G}s_{k,h}k\right)h\in(S*G)^{G'}G.\end{aligned} \tag{4.3}$$

Hence, for every $s \in S^P$ and every $g \in G$, $sg \in (S*G)^{G'}GG = (S*G)^{G'}G$, that is $S^PG \subseteq (S*G)^{G'}G$.

On the other hand, for any $\sum_{k\in G} s_k k \in (S*G)^{G'}$, we have

$$\sum_{k\in G} s_k k = g\sum_{k\in G} s_k k g^{-1} = \sum_{k\in G} g(s_k)gkg^{-1} \quad \text{for every } g \in G. \tag{4.4}$$

Therefore, $g(s_k) = s_{gkg^{-1}}$ for every $g \in G$ since $\{k \mid k \in G\}$ is a basis of $S*G$ over S. In particular, for every $p \in P$, $p(s_k) = s_{pkp^{-1}} = s_k$, i.e., $s_k \in S^P$ for every $k \in G$ and, therefore, $\sum_{k\in G} s_k k \in S^PG$ if $\sum_{k\in G} s_k k \in (S*G)^{G'}$. Hence, $(S*G)^{G'} \subseteq S^PG$. Therefore, $(S*G)^{G'}G \subseteq S^PGG = S^PG$. Hence, $S^PG = (S*G)^{G'}G$. So, $(S*G)^{P'} = S^PG = (S*G)^{G'}G = (S*G)^{G'}ZG$. Consequently, by Lemma 4.2, ZG is a central (G'/P')-Galois algebra. □

Acknowledgement. This work was supported by a Caterpillar Fellowship at Bradley University. We would like to thank Caterpillar Inc. for the support.

References

[1] R. Alfaro and G. Szeto, *The centralizer on H-separable skew group rings*, Rings, Extensions, and Cohomology (Evanston, IL, 1993), Dekker, New York, 1994, pp. 1–7. MR 95g:16027. Zbl 812.16038.

[2] ______, *Skew group rings which are Azumaya*, Comm. Algebra **23** (1995), no. 6, 2255–2261. MR 96b:16027. Zbl 828.16030.

[3] ______, *On Galois extensions of an Azumaya algebra*, Comm. Algebra **25** (1997), no. 6, 1873–1882. MR 98h:13007. Zbl 890.16017.

[4] F. DeMeyer and E. Ingraham, *Separable Algebras over Commutative Rings*, vol. 181, Springer-Verlag, Berlin, 1971. MR 43#6199. Zbl 215.36602.

[5] F. R. DeMeyer, *Some notes on the general Galois theory of rings*, Osaka J. Math. **2** (1965), 117–127. MR 32#128. Zbl 143.05602.

[6] K. Sugano, *Note on semisimple extensions and separable extensions*, Osaka J. Math. **4** (1967), 265–270. MR 37#1412. Zbl 199.07901.

[7] ______, *On a special type of Galois extensions*, Hokkaido Math. J. **9** (1980), no. 2, 123–128. MR 82c:16036. Zbl 467.16005.

Szeto and Xue: Department of Mathematics, Bradley University, Peoria, Illinois 61625, USA