Internat. J. Math. & Math. Sci.
Vol. 24, No. 1 (2000) 59–61
S0161171200003537

ON AN ISOLATION AND A GENERALIZATION OF HÖLDER'S INEQUALITY

XIAOJING YANG

(Received 23 December 1998 and in revised form 16 June 1999)

ABSTRACT. We generalize the well-known Hölder inequality and give a condition at which the equality holds.

Keywords and phrases. Hölder inequality, equality.

2000 Mathematics Subject Classification. Primary 34C10; Secondary 34L05.

Let us consider the famous Hölder inequality [1]

$$\sum_{i=1}^{m}\prod_{j=1}^{n}X_{ij} \le \prod_{j=1}^{n}\left(\sum_{i=1}^{m}X_{ij}^{pj}\right)^{1/pj}, \tag{1}$$

where $X_{ij} > 0$ $(1 \le i \le m,\ 1 \le j \le n)$, $p_j > 1$, $\sum_{j=1}^{n}(1/p_j) = 1$.

We give a generalization of (1), first, we need the following lemmas.

LEMMA 1. *Let $y_i = \prod_{j=1}^{n} X_{ij}$, $i = 1,2,\dots,m$, then $\sum_{k=1}^{n} 1/p_k(\sum_{i=1}^{m} y_i \ln z_{ik}) = 0$, where $z_{ik} = X_{ik}^{pk}/y_i$, $i = 1,2,\dots,m$.*

PROOF. Since $y_i = \prod_{j=1}^{n} X_{ij}$, we have $\ln y_i = \sum_{j=i}^{n} \ln X_{ij}$ and

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{p_k}\ln z_{ik} &= \sum_{k=1}^{n}\frac{1}{p_k}\ln\left(\frac{X_{ik}^{pk}}{y_i}\right) = \sum_{k=1}^{n}\frac{1}{p_k}\left(p_k\ln X_{ik} - \ln y_i\right)\\ &= \sum_{k=1}^{n}\ln X_{ik} - \sum_{k=1}^{n}\frac{1}{p_k}\ln y_i = \ln\prod_{k=1}^{n}X_{ik} - \ln y_i = 0, \quad i = 1,2,\dots,m\end{aligned} \tag{2}$$

here we have used the fact $\sum_{k=1}^{n}(1/p_k) = 1$, from the above calculations we obtain

$$\left(\sum_{i=1}^{m} y_i\right)\left(\sum_{k=1}^{n}\frac{1}{p_k}\ln z_{ik}\right) = \sum_{k=1}^{n}\frac{1}{p_k}\left(\sum_{i=1}^{m} y_i \ln z_{ik}\right) = 0. \tag{3}$$

□

LEMMA 2. *If a, b are positive numbers, then*

$$\begin{aligned}(\ln a - \ln b)(a^t - b^t) &\ge 0, \quad \text{if } t \ge 0,\\ (\ln a - \ln b)(a^t - b^t) &\le 0, \quad \text{if } t \le 0,\end{aligned} \tag{4}$$

and the equalities hold if and only if $(a-b)t = 0$.

Proof. If $a = b$, (4) holds, let $a \neq b$, by the mean value theorem $f(a) - f(b) = (a-b)f'(z)$ with $z \in (a,b)$, we have

$$\begin{aligned} \ln a - \ln b &= (a-b)\frac{1}{z_1}, \quad z_1 \in (a,b), \\ (a^t - b^t) &= (a-b)tz_2^{t-1}, \quad z_2 \in (a,b) \end{aligned} \tag{5}$$

hence

$$(\ln a - \ln b)(a^t - b^t) = \frac{(a-b)^2 t z_2^{t-1}}{z_1} \begin{cases} \geq 0, & \text{if } t \geq 0 \\ \leq 0, & \text{if } t \leq 0 \end{cases} \tag{6}$$

and the equality holds if and only if $t = 0$. □

The main result of our paper is the following theorem.

Theorem 3. *If $X_{ij} > 0$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$, $p_i > 1$, $\sum_{i=1}^n (1/p_i) = 1$, let*

$$h(t) = \prod_{k=1}^n \left[\sum_{i=1}^m y_i \left(\frac{X_{ik}^{p_k}}{y_i} \right)^t \right]^{1/p_k} = \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij} \right)^{(1-t)} (X_{ik}^{p_k})^t \right]^{1/p_k}. \tag{7}$$

Then

$$\begin{aligned} h'(t) &\geq 0, \quad \text{if } t \geq 0, \\ h'(t) &\leq 0, \quad \text{if } t \leq 0, \end{aligned} \tag{8}$$

the equality holds if and only if $t = 0$, or $X_{ik}^{p_k} / \prod_{\tau=1}^n X_{i\tau} = X_{jk}^{p_k} / \prod_{\tau=1}^n X_{j\tau}$ for $1 \leq i, j \leq m$, $k = 1,2,\ldots,n$, that is, $X_{1k}^{p_k} / \prod_{j=1}^n X_{1j} = X_{2k}^{p_k} / \prod_{j=1}^n X_{2j} = \cdots = X_{mk}^{p_k} / \prod_{j=1}^n X_{mj}$.

Proof. Let $H(t) = \ln h(t)$, $t \in \mathbb{R}$, then by Lemmas 1 and 2 we obtain

$$\begin{aligned} H'(t) = \frac{h'(t)}{h(t)} &= \sum_{k=1}^n \frac{1}{p_k} \frac{\left(\sum_{i=1}^m y_i z_{ik}^t \ln z_{ik}\right)}{\left(\sum_{i=1}^m y_i z_{ik}^t\right)} \\ &= \sum_{k=1}^n \frac{1}{p_k} \frac{\left(\sum_{i=1}^m y_i z_{ik}^t \ln z_{ik}\right)}{\left(\sum_{i=1}^m y_i z_{ik}^t\right)} - \sum_{k=1}^n \frac{1}{p_k} \frac{\left(\sum_{i=1}^m y_i \ln z_{ik}\right)}{\left(\sum_{i=1}^m y_i\right)} \\ &= \sum_{k=1}^n \frac{1}{p_k} \frac{\sum_{1 \leq i < j \leq m} y_i y_j (\ln z_{ik} - \ln_{jk})(z_{ik}^t - z_{jk}^t)}{\left(\sum_{i=1}^m y_i\right)\left(\sum_{i=1}^m y_i z_{ik}^t\right)} \begin{cases} \geq 0, & \text{if } t \geq 0, \\ \leq 0, & \text{if } t \leq 0. \end{cases} \end{aligned} \tag{9}$$

The equality holds if and only if $t = 0$ or

$$z_{1k} = z_{2k} = \cdots = z_{mk}, \quad k = 1,2,\ldots,n, \tag{10}$$

where $y_i = \prod_{j=1}^n X_{ij}$, $z_{ik} = X_{ik}^{p_k} / \prod_{j=1}^n X_{ij}$, $i = 1,2,\ldots,m$, $k = 1,2,\ldots,n$. This completes the proof . □

Corollary 4. *Let $t_2 < t_1 \leq 0$ or $0 \leq t_1 < t_2$, we have*

$$\prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij} \right)^{1-t_1} X_{ik}^{p_k t_1} \right]^{1/p_k} \leq \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij} \right)^{1-t_2} X_{ik}^{p_k t_2} \right]^{1/p_k}, \tag{1'}$$

equality holds if and only if for any $k = 1, 2, \ldots, n$,

$$\frac{X_{ik}^{p_k}}{\prod_{\tau=1}^{n} X_{i\tau}} = \frac{X_{jk}^{p_k}}{\prod_{\tau=1}^{n} X_{j\tau}}, \quad \textit{for } 1 \le i \le j \le m, \tag{2'}$$

and for all $0 < t < 1$,

$$\sum_{i=1}^{m}\prod_{j=1}^{n} X_{ij} \le \prod_{k=1}^{n}\left[\sum_{i=1}^{m}\left(\prod_{j=1}^{i} X_{ij}\right)^{1-t} X_{ik}^{p_k t}\right]^{1/p_k} \le \prod_{i=1}^{n}\left(\sum_{j=1}^{m} X_{ij}^{p_j}\right)^{1/p_j} \tag{3'}$$

equality holds only $(2')$ *holds.*

References

[1] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin, 1961. MR 28#1266. Zbl 097.26502.

Xiaojing Yang: Department of Mathematics, Tsinghua University, Beijing, China