

## THE COXETER GROUP $D_n$

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**ABSTRACT.** We show that the Coxeter group  $D_n$  is the split extension of  $n - 1$  copies of  $Z_2$  by  $S_n$  for a given action of  $S_n$  described in the paper. We also find the centre of  $D_n$  and some of its other important subgroups.

**Keywords and phrases.** Coxeter group, split extension.

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**1. Introduction.** The Coxeter group  $D_n$  [5] has the presentation

$$\begin{aligned}
 D_n = \langle x_1, x_2, \dots, x_n \mid & x_i^2 = e, 1 \leq i \leq n; (x_i x_{i+1})^3 = e, 2 \leq i \leq n-1; \\
 & (x_i x_j)^2 = e, |i-j| \neq 1 \text{ and } (i, j) \neq (1, 3); (x_1 x_2)^2 = (x_1 x_3)^3 = e \rangle.
 \end{aligned}
 \tag{1.1}$$

and the graph given in Figure 1.1.

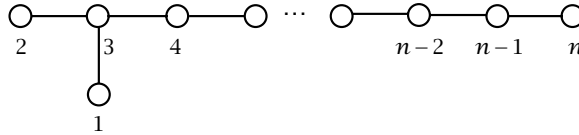


FIGURE 1.1.

In [1], we have shown that  $D_4$  is solvable with derived length 4 and that its order is 192. In this paper, we explain the algebraic structure of  $D_n$  and find its centre. We also find the derived series; and the growth series of  $D_n$  for  $4 \leq n \leq 8$ .

**2. The structure of  $D_n$ .** In [4], we have shown that the Coxeter group  $B_n$  whose graph is given in Figure 2.1

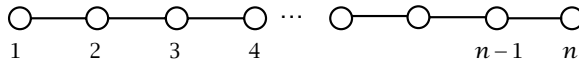


FIGURE 2.1.

is the wreath product of  $Z_2$  by  $S_n$ , that is,  $B_n$  is the split extension of  $Z_2^n$  by  $S_n$ . Let  $Z_2^n$  have the presentation

$$H = Z_2^n = \langle a_1, a_2, \dots, a_n \mid a_i^2 = e, 1 \leq i \leq n; (a_i a_j)^2 = e, 1 \leq i < j \leq n \rangle. \tag{2.1}$$

Let  $K$  be the even subgroup of  $H$ , that is, the subgroup consisting of even words in  $H$ . It is easy to find the following presentation for  $K$ :

$$K = \langle b_2, b_3, \dots, b_n \mid b_i^2 = e, 2 \leq i \leq n; (b_i b_j)^2 = e, 2 \leq i < j \leq n \rangle, \tag{2.2}$$

where  $b_i = a_1 a_i$ ,  $2 \leq i \leq n$ . Thus  $K$  is  $Z_2^{n-1}$ . In the extension  $B_n \cong Z_2^n \rtimes S_n$ , the action of  $S_n$  on  $Z_2^n$  is a natural one that can be explained as follows. Let  $S_n$  have the presentation

$$S_n = \langle x_1, x_2, \dots, x_{n-1} \mid x_i^2 = e, 1 \leq i \leq n-1; (x_i x_{i+1})^3 = e, 1 \leq i \leq n-2; (x_i x_j)^2 = e, 1 \leq i < j-1 \leq n-2 \rangle, \tag{2.3}$$

where  $x_i$  is the transposition  $(i \ i+1)$ . The action of  $S_n$  on  $H$  is given by

$$(a_1, a_2, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n)^{x_i} = (a_1, a_2, \dots, a_{i-1}, a_{i+1}, a_i, \dots, a_n). \tag{2.4}$$

Using this action we compute the action of  $S_n$  on  $K$  as follows:

$$\begin{aligned} (b_2, b_3, \dots, b_n)^{x_1} &= (a_1 a_2, a_1 a_3, \dots, a_1 a_n)^{x_1} \\ &= (a_2 a_1, a_2 a_3, \dots, a_2 a_n) \\ &= (b_2^{-1}, b_2^{-1} b_3, \dots, b_2^{-1} b_n), \\ (b_2, b_3, \dots, b_i, b_{i+1}, \dots, b_n)^{x_i} &= (a_1 a_2, \dots, a_1 a_i, a_1 a_{i+1}, \dots, a_1 a_n)^{x_i} \\ &= (a_1 a_2, \dots, a_1 a_{i+1}, a_1 a_i, \dots, a_1 a_n) \\ &= (b_2, \dots, b_{i+1}, b_i, \dots, b_n), \quad 2 \leq i \leq n-1. \end{aligned} \tag{2.5}$$

We use this action to construct a split extension  $E$  of  $K = Z_2^{n-1}$  by  $S_n$ . A presentation for this extension is given by the method in [2],  $E = \langle \text{generators of } K, \text{ generators of } S_n \mid \text{relations of } K, \text{ relations of } S_n, \text{ the action of } S_n \text{ on } K \rangle$ .

We change the action of  $S_n$  on  $K$  to the following relations:

$$b_2^{x_1} = b_2^{-1}, \tag{2.6}$$

$$b_i^{x_1} = b_2^{-1} b_i, \quad 3 \leq i \leq n, \tag{2.7}$$

$$b_i^{x_i} = b_{i+1}, \quad 2 \leq i \leq n-1, \tag{2.8}$$

$$b_{i+1}^{x_i} = b_i, \quad 2 \leq i \leq n-1, \tag{2.9}$$

$$b_j^{x_i} = b_j, \quad 2 \leq j \leq n, \quad 2 \leq i \leq n-1, \quad j \neq i, \quad j \neq i+1. \tag{2.10}$$

We will use Tietze transformations to show that  $E$  is isomorphic to  $D_n$ . But before that we observe the following. The relation (2.8) implies that  $b_i = b_2^{x_2 x_3 \cdots x_{i-1}}$ ,  $3 \leq i \leq n$ . Let  $u_i = x_2 x_3 \cdots x_i$ .

**LEMMA 2.1.** *The following identities hold in the group  $S_{n-1}$ :*

- (i)  $u_k x_i = x_{i+1} u_k$ , if  $2 \leq i \leq k$ ,
- (ii)  $u_k x_i = u_{k-1}$ , if  $i = k$ ,
- (iii)  $u_k x_i = u_{k+1}$ , if  $i = k+1$ ,
- (iv)  $u_k x_i = x_i u_k$ , if  $i > k+1$ ,
- (v)  $u_k u_i = (x_3 x_4 \cdots x_{i+1}) u_k$ , if  $2 \leq i < k$ ,
- (vi)  $u_k u_i = (x_3 x_4 \cdots x_i) u_{k-1}$ , if  $i \geq k$ .

**PROOF.** We will make use of the relations of  $S_{n-1}$ .

(i)

$$\begin{aligned}
 u_k x_i &= (x_2 x_3 \cdots x_i x_{i+1} \cdots x_k) x_i \\
 &= x_2 x_3 \cdots x_i x_{i+1} x_i \cdots x_k \\
 &= x_2 x_3 \cdots x_{i+1} x_i x_{i+1} \cdots x_k \\
 &= x_{i+1} x_2 x_3 \cdots x_i x_k = x_{i+1} u_k.
 \end{aligned}
 \tag{2.11}$$

(ii), (iii), and (iv) are clear, while (v) and (vi) are applications of (i) to (iv). □

We reduce relation (2.6) to (2.10) as follows. Relation (2.6) easily becomes

$$(x_1 b_2)^2 = e. \tag{2.12}$$

Using Lemma 2.1, (2.7) becomes

$$(b_2 x_1 x_2)^3 = e. \tag{2.13}$$

Using  $b_i = b_2^{x_2 x_3 \cdots x_{i-1}}$ , (2.8) becomes redundant. Relation (2.9), using Lemma 2.1, becomes redundant. Using Lemma 2.1, (2.10) becomes

$$(x_i b_2)^2 = e, \quad 3 \leq i \leq n. \tag{2.14}$$

The relation  $b_i^2 = e = (b_i b_j)^2$  become redundant for  $i \geq 3$ . Let  $c = x_1 b_2$ . Then (2.12) becomes  $c^2 = e$ . Relation (2.13) becomes  $(c x_2)^3 = e$ . Relation (2.14) becomes  $(c x_i)^2 = e$  for  $i \geq 3$ . The relation  $b_2^2 = e$  becomes  $(c x_1)^2 = e$ . Therefore a presentation for  $E$  is

$$\begin{aligned}
 E = \langle x_1, x_2, \dots, x_{n-1}, c \mid &x_i^2 = e, 1 \leq i \leq n-1; c^2 = e; \\
 &(x_i x_j)^2 = e, 1 \leq i < j-1 \leq n-2; (x_i x_{i+1})^3 = e, 1 \leq i \leq n-2; \\
 &(c x_2)^3 = e; (c x_i)^2 = e, 1 \leq i \leq n-1 \text{ and } i \neq 2 \rangle.
 \end{aligned}
 \tag{2.15}$$

Consequently, we have proved the following theorem.

**THEOREM 2.2.** *The group  $D_n$  is the split extension of  $n-1$  copies of  $Z_2$  by  $S_n$ .*

**REMARK 2.3.** Let us describe the relation between the Coxeter groups  $B_n$  and  $D_n$ . It is easy to show that  $D_n$  is the subgroup of  $B_n$  consisting of all elements of even length in  $B_n$ . Thus  $D_n$  is a subgroup of  $B_n$  of index 2. On the other hand, let  $S_n$  be the symmetric group of degree  $n$  generated by  $y_i, 1 \leq i \leq n-1$ . We consider the map  $\theta : D_n \rightarrow S_n$  defined by  $\theta(x_1) = y_2, \theta(x_2) = y_2$ , and  $\theta(x_i) = y_{i-1}, 3 \leq i \leq n$ . Using the Reidemeister-Schreier process, it is possible to show that  $\theta^{-1}(S_{n-1}) \cong B_{n-1}$ . Hence  $B_{n-1}$  is a subgroup of  $D_n$  of index  $n$ . We observe the graph given in Figure 2.2.

The orders of  $B_n$  and  $D_n$  are  $|B_n| = 2^n n!$  and  $|D_n| = 2^{n-1} n!$ , respectively. It is also easy to see that  $B'_n \cong D'_n$ .

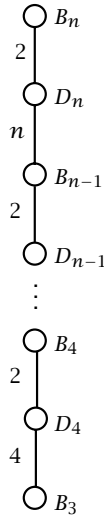


FIGURE 2.2.

**3. The derived series of  $D_n$ .** In [1], we showed that  $D_4$  is solvable of derived length 4. For  $n > 4$  we use the Reidemeister-Shreier process to find the following presentation for  $D'_n$ :

$$\begin{aligned}
 D'_n = \langle b_2, b_3, \dots, b_n \mid & b_2^2 = b_3^3 = b_i^2 = e, 4 \leq i \leq n; \\
 & (b_i b_{i+1})^3 = e, 2 \leq i \leq n-1; (b_i b_j)^2 = e, 2 \leq i < j-1 \leq n-1 \rangle.
 \end{aligned}
 \tag{3.1}$$

The group  $D'_n/D''_n$  is trivial. Hence  $D'_n$  is a complete group. We thus have the following theorem.

**THEOREM 3.1.**  *$D_n$  is solvable of derived length 4 if  $n = 4$ . If  $n > 4$ , then  $D_n$  is not solvable.*

**4. The centre of  $D_n$ .** We use the structure of  $D_n$  explained in Section 2 to prove the following theorem.

**THEOREM 4.1.** *The centre of  $D_n$  is  $Z_2$  if  $n$  is even and trivial if  $n$  is odd.*

**PROOF.** In Section 2, we showed that  $D_n$  is the split extension of  $Z_2^{n-1}$  by  $S_n$ . This means the existence of an epimorphism  $\theta : D_n \rightarrow S_n$ , where  $\ker \theta = Z_2^{n-1}$ . It follows that  $\theta(Z(D_n)) \subseteq Z(\theta(D_n)) = Z(S_n) = \{e\}$ . Hence  $Z(D_n) \subseteq \ker \theta = Z_2^{n-1}$ . We use the previous notation where  $S_n = \langle x_1, x_2, \dots, x_{n-1} \rangle$  and  $Z_2^{n-1} = \langle b_2, b_3, \dots, b_n \rangle$  and the previous action. We let  $w \in Z(D_n) \Rightarrow w \in Z_2^{n-1}$  and  $w^{x_i} = w$  for  $1 \leq i \leq n-1$ . Also  $w = b_2^{\epsilon_2} b_3^{\epsilon_3} \dots b_n^{\epsilon_n}$ , where  $\epsilon_i = 0$  or  $1$  since  $b_i^2 = e$ . Using the action of  $S_n$  on  $Z_2^{n-1}$ , we get

$$\left( b_2^{\epsilon_2} b_3^{\epsilon_3} \dots b_i^{\epsilon_i} b_{i+1}^{\epsilon_{i+1}} \dots b_n^{\epsilon_n} \right)^{x_i} = b_2^{\epsilon_2} b_3^{\epsilon_3} \dots b_i^{\epsilon_i} b_{i+1}^{\epsilon_{i+1}} \dots b_n^{\epsilon_n}.
 \tag{4.1}$$

Letting  $2 \leq i \leq n - 1$ , we get

$$b_2^{\epsilon_2} b_3^{\epsilon_3} \cdots b_{i+1}^{\epsilon_i} b_i^{\epsilon_{i+1}} \cdots b_n^{\epsilon_n} = b_2^{\epsilon_2} b_3^{\epsilon_3} \cdots b_i^{\epsilon_i} b_{i+1}^{\epsilon_{i+1}} \cdots b_n^{\epsilon_n}, \tag{4.2}$$

and so  $b_{i+1}^{\epsilon_i} b_i^{\epsilon_{i+1}} = b_i^{\epsilon_i} b_{i+1}^{\epsilon_{i+1}}$ . This implies  $\epsilon_i = \epsilon_{i+1}$  and so  $\epsilon_2 = \epsilon_3 = \cdots = \epsilon_n$ . Hence  $w = b_2 b_3 \cdots b_n$  or  $w = b_2^0 b_3^0 \cdots b_n^0 = e$ . Now, we consider the action of  $x_1$  on  $w$  in the following two cases:

(a) If  $n$  is even, we get  $(b_2 b_3 \cdots b_n)^{x_1} = b_2^{n-2} b_2 b_3 \cdots b_n = b_2 b_3 \cdots b_n$  since  $b_2^2 = e$ . Hence  $b_2 b_3 \cdots b_n$  is in the centre of  $D_n$ . Since  $(b_2 b_3 \cdots b_n)^2 = e$ , we get  $Z(D_n) = Z_2$ .

(b) If  $n$  is odd,  $(b_2 b_3 \cdots b_n)^{x_1} = b_3 b_4 \cdots b_n$  and  $b_2 b_3 \cdots b_n$  does not commute with  $x_1$ . Thus  $w$  is  $b_2^0 b_3^0 \cdots b_n^0 = e$  and  $Z(D_n) = \{e\}$ . □

**5. The growth series.** The growth series, in the sense of Milnor and Gromov, of  $D_n$  for  $4 \leq n \leq 8$  were computed as follows [3]:

$$y(D_4) = (1+t)^4(1+t^2)^2(1-t+t^2)(1+t+t^2), \tag{5.1}$$

$$y(D_5) = (1+t)^4(1+t^2)^2(1+t^4)(1-t+t^2)(1+t+t^2)(1+t+t^2+t^3+t^4), \tag{5.2}$$

$$y(D_6) = (1+t)^6(1+t^2)^2(1+t^4)(1-t+t^2)^2(1+t+t^2)^2 \times (1-t+t^2-t^3+t^4)(1+t+t^2+t^3+t^4), \tag{5.3}$$

$$y(D_7) = (1+t)^6(1+t^2)^3(1+t^4)(1-t+t^2)^2(1+t+t^2)^2(1-t^2+t^4) \times (1-t+t^2-t^3+t^4)(1+t+t^2+t^3+t^4)(1+t+t^2+t^3+t^4+t^5+t^6), \tag{5.4}$$

$$y(D_8) = (1+t)^8(1+t^2)^4(1+t^4)^2(1-t+t^2)^2(1+t+t^2)^2 \times (1-t^2+t^4)(1-t+t^2-t^3+t^4)(1+t+t^2+t^3+t^4) \times (1-t+t^2-t^3+t^4-t^5+t^6)(1+t+t^2+t^3+t^4+t^5+t^6). \tag{5.5}$$

We make two observations about these growth polynomials. First, each growth series is a product of cyclotomic polynomials. Second, the value of the series at 1 is the order of the corresponding group and the degree of the growth series equals the length of the element of maximal length.

We have not yet succeeded in finding the growth series of  $D_n$  for general  $n$ .

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**REFERENCES**

[1] M. A. Albar, *Analogues of the braid group and their corresponding Coxeter groups*, Comm. Algebra **12** (1984), no. 23-24, 2977-2984. MR 86g:20041. Zbl 591.20046.  
 [2] ———, *On presentation of group extensions*, Comm. Algebra **12** (1984), no. 23-24, 2967-2975. MR 86g:20040. Zbl 551.20017.  
 [3] M. A. Albar, M. A. Al-Hamed, and N. A. Al-Saleh, *The growth of Coxeter groups*, Math. Japon. **47** (1998), no. 3, 417-428. MR 99f:20066. Zbl 912.20032.  
 [4] M. A. Albar and N. A. Al-Saleh, *On the affine Weyl group of type  $B_n$* , Math. Japon. **35** (1990), no. 4, 599-602. MR 91d:20030. Zbl 790.20048.

- [5] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, 4th ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*, vol. 14, Springer-Verlag, Berlin, 1980. MR 81a:20001. Zbl 422.20001.

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