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ABEL-TYPE WEIGHTED MEANS TRANSFORMATIONS INTO ℓ

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ABSTRACT. Let $q_k = \binom{k+\alpha}{k}$ for $\alpha > -1$ and $Q_n = \Sigma_{k=0}^{n} q_k$. Suppose $A_q = \{a_{nk}\}$, where $a_{nk} = q_k/Q_n$ for $0 \le k \le n$ and 0 otherwise. A_q is called the Abel-type weighted mean matrix. The purpose of this paper is to study these transformations as mappings into ℓ. A necessary and sufficient condition for A_q to be ℓ-ℓ is proved. Also some other properties of the A_q matrix are investigated.

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1. Introduction. Throughout this paper, we assume that $\alpha > -1$ and Q_n is the partial sums of the sequence $\{q_k\}$, where q_k is as above. Let $A_q = \{a_{nk}\}$. Then the Abel-type weighted mean matrix, denoted by A_q, is defined by

$$a_{nk} = \begin{cases} \dfrac{q_k}{Q_n} & \text{for } 0 \le k \le n, \\ 0 & \text{for } k > n. \end{cases} \tag{1.1}$$

The A_q matrix is the weighted mean matrix that is associated with the Abel-type matrix introduced by M. Lemma in [5]. It is regular, indeed, totally regular.

2. Basic notation and definitions. Let $A = (a_{nk})$ be an infinite matrix defining a sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \tag{2.1}$$

where $(Ax)_n$ denotes the nth term of the image sequence Ax. Let y be a complex number sequence. Throughout this paper, we use the following basic notation and definitions:

(i) $c = \{\text{The set of all convergent complex sequences}\}$,
(ii) $\ell = \{y : \sum_{k=0}^{\infty} |y_k| < \infty\}$,
(iii) $\ell^P = \{y : \sum_{k=0}^{\infty} |y_k|^P < \infty\}$,
(iv) $\ell(A) = \{y : Ay \in \ell\}$,
(v) $G = \{y : y_k = O(r^k) \text{ for some } r \in (0,1)\}$,
(vi) $G_w = \{y : y_k = O(r^k) \text{ for some } r \in (0,w),\ 0 < w < 1\}$.

Definition 1. If X and Y are sets of complex number sequences, then the matrix A is called an X-Y matrix if the image Au of u under the transformation A is in Y, whenever u is in X.

3. Some basic facts. The following facts are used repeatedly.

(1) For any real number $\alpha > -1$ and any nonnegative integer k, we have

$$\binom{k+\alpha}{k} \sim \frac{k^\alpha}{\Gamma(\alpha+1)} \quad (\text{as } k \longrightarrow \infty). \tag{3.1}$$

(2) For any real number $\alpha > -1$, we have

$$\sum_{k=0}^{n} \binom{k+\alpha}{k} = \binom{n+\alpha+1}{n}. \tag{3.2}$$

(3) Suppose $\{a_n\}$ is sequence of nonnegative numbers with $a_0 > 0$, that

$$A_n = \sum_{k=0}^{n} a_k \longrightarrow \infty. \tag{3.3}$$

Let

$$a(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad A(x) = \sum_{k=0}^{\infty} A_k x^k, \tag{3.4}$$

and suppose that

$$a(x) < \infty \quad \text{for } 0 < x < 1. \tag{3.5}$$

Then it follows that

$$(1-x)A(x) = a(x) \quad \text{for } 0 < x < 1. \tag{3.6}$$

4. The main results

Lemma 1. *If A_q is an ℓ-ℓ matrix, then $1/Q \in \ell$.*

Proof. By the Knopp-Lorentz theorem [4], A_q is an ℓ-ℓ matrix implies that

$$\sum_{k=0}^{\infty} |a_{n,0}| < \infty, \tag{4.1}$$

and consequently we have $1/Q \in \ell$. □

Lemma 2. *We have that $1/Q \in \ell$ if and only if $\alpha > 0$.*

Proof. By using (3.1), we have

$$\frac{1}{Q_n} \sim \frac{\Gamma(\alpha+2)}{n^{\alpha+1}} \tag{4.2}$$

and hence the assertion easily follows. □

Lemma 3. *If $1/Q \in \ell$, then A_q is an ℓ-ℓ matrix.*

Proof. By Lemma 2, we have $\alpha > 0$. To show that A_q is an ℓ-ℓ matrix, we must show that the condition of the Knopp-Lorentz theorem [4] holds. Using (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} |a_{nk}| &= \binom{k+\alpha}{k} \sum_{n=k}^{\infty} \frac{1}{Q_n} = \binom{k+\alpha}{k} \sum_{n=k}^{\infty} \frac{1}{\binom{n+\alpha+1}{n}} \\ &\le M_1 K^{\alpha} \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+1}} \quad \text{for some } M_1 > 0, \\ &\le M_1 M_2 k^{\alpha} \int_k^{\infty} \frac{dx}{x^{\alpha+1}} \quad \text{for some } M_2 > 0, \\ &= \frac{M_1 M_2}{\alpha}.\end{aligned} \tag{4.3}$$

Hence, by the Knopp-Lorentz theorem [4], A_q is an ℓ-ℓ matrix. □

Theorem 1. *The following statements are equivalent:*
(1) *A_q is an ℓ-ℓ matrix;*
(2) *$1/Q \in \ell$;*
(3) *$\alpha > 0$.*

Proof. The theorem easily follows by Lemmas 1, 2, and 3. □

Remark 1. In Theorem 1, we showed that A_q is an ℓ-ℓ matrix if and only if $1/Q \in \ell$. But the converse is not true in general for any weighted mean matrix W_p that corresponds to a sequence-to-sequence variant of the general J_p power series method of summability [1]. To see this, let

$$p_k = \left(\ln(k+2)\right)^{\alpha}, \quad \alpha > 1. \tag{4.4}$$

We show that $1/P \in \ell$ but W_p is not an ℓ-ℓ matrix. We have

$$\begin{aligned}P_n &= \sum_{k=0}^{n} \left(\ln(k+2)\right)^{\alpha} \\ &\sim \int_0^n \left(\ln(x+2)\right)^{\alpha} dx \quad \text{(by [6, Thm. 1.20])} \\ &\sim (n+2)\left(\ln(n+2)\right)^{\alpha},\end{aligned} \tag{4.5}$$

using integration by parts repeatedly. This yields

$$\frac{1}{P_n} \sim \frac{1}{(n+2)\left(\ln(n+2)\right)^{\alpha}} \tag{4.6}$$

and by the condensation test, it follows that $1/P \in \ell$.

Next, we show that W_p is not an ℓ-ℓ matrix by showing that the condition of the Knopp-Lorentz theorem [4] fails to hold. Using (4.6), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}|a_{nk}| &= (\ln(k+2))^{\alpha}\sum_{n=k}^{\infty}\frac{1}{P_n} \\ &\geq M_1(\ln(k+2))^{\alpha}\sum_{n=k}^{\infty}\frac{1}{(n+2)(\ln(n+2))^{\alpha}} \quad \text{for some } M_1>0 \\ &\geq M_1M_2(\ln(k+2))^{\alpha}\int_k^{\infty}\frac{dx}{(x+2)(\ln(x+2))^{\alpha}} \quad \text{for some } M_2>0 \\ &= \frac{M_1M_2}{\alpha-1}(\ln(k+2)).\end{aligned} \tag{4.7}$$

Thus, we have

$$\sup_k\left\{\sum_{n=0}^{\infty}a_{nk}\right\}=\infty, \tag{4.8}$$

and hence W_p is not an ℓ-ℓ matrix. □

Corollary 1. *A_Q is an ℓ-ℓ matrix.*

Proof. Since $Q_n=\binom{n+\alpha+1}{n}$ and $\alpha>-1$ implies that $\alpha+1>0$, the assertion easily follows by Theorem 1. □

Corollary 2. *A_q is an ℓ-ℓ matrix if and only if* $\lim_n(Q_n/nq_n)<1$.

Proof. By Theorem 1, A_q is an ℓ-ℓ matrix implies that $\alpha>0$, and as a consequence we have $1/(\alpha+1)<1$. Now using (3.1), we have

$$\lim_n\left(\frac{Q_n}{nq_n}\right)=\lim_n\frac{n^{\alpha+1}\Gamma(\alpha+1)}{\Gamma(\alpha+2)n^{\alpha+1}}=\frac{1}{\alpha+1}<1. \tag{4.9}$$

Conversely, if $\lim_n(Q_n/nq_n)<1$, then it follows from (4.9) that $1/(\alpha+1)<1$ and consequently we have $\alpha>0$, and hence, by Theorem 1, A_q is an ℓ-ℓ matrix. □

Corollary 3. *Suppose that $z_k=\binom{k+\beta}{k}$ and $\alpha<\beta$; then A_z is an ℓ-ℓ matrix whenever A_q is an ℓ-ℓ matrix.*

Proof. The corollary follows easily by Theorem 1. □

Lemma 4. *If the Abel-type matrix $A_{\alpha,t}$* [5] *is an ℓ-ℓ matrix, then $A_{\alpha+1,t}$ is also an ℓ-ℓ matrix.*

Proof. By the Knopp-Lorentz theorem [4], $A_{\alpha,t}$ is an ℓ-ℓ matrix implies that

$$\sup_k\left\{\sum_{n=0}^{\infty}|a_{nk}|\right\}<\infty. \tag{4.10}$$

This is equivalent to

$$\sup_k\left\{\binom{k+\alpha}{k}\sum_{n=0}^{\infty}t_n^k(1-t_n)^{\alpha+1}\right\}<\infty. \tag{4.11}$$

Now from (4.11), we can easily conclude that

$$\sup_k \left\{ \binom{k+\alpha+1}{k} \sum_{n=0}^{\infty} t_n^k (1-t_n)^{\alpha+2} \right\} < \infty. \tag{4.12}$$

Hence, $A_{\alpha+1,t}$ is an ℓ-ℓ matrix. □

The next theorem compares the summability fields of the matrices A_q and $A_{\alpha,t}$ [5].

Theorem 2. *If $A_{\alpha,t}$ and A_q are ℓ-ℓ matrices, then $\ell(A_q) \subseteq \ell(A_{\alpha,t})$.*

Proof. Let $x \in \ell(A_q)$. Then we show that $x \in \ell(A_{\alpha,t})$. Let y be the A_q-transform of the sequence x. Then we have

$$y_n Q_n = \sum_{k=0}^{n} q_k x_k. \tag{4.13}$$

Now since $y_n Q_n$ is the partial sums of the sequence q_x, using (3.6) it follows that

$$(1-t_n) \sum_{k=0}^{\infty} Q_k y_k t_n^k = \sum_{k=0}^{\infty} q_k x_k t_n^k. \tag{4.14}$$

This yields

$$(1-t_n)^{\alpha+2} \sum_{k=0}^{\infty} Q_k y_k t_n^k = (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} q_k x_k t_n^k, \tag{4.15}$$

and as a consequence we have $(A_{\alpha+1,t}y)_n = (A_{\alpha,t}x)_n$. By Lemma 4, $A_{\alpha,t}$ is an ℓ-ℓ matrix implies that $A_{\alpha+1,t}$ is also an ℓ-ℓ matrix, and from the assumption that $x \in \ell(A_q)$, it follows that $y \in \ell$. Consequently, we have $A_{\alpha+1,t}y \in \ell$ and this is equivalent to $A_{\alpha,t}x \in \ell$. Thus, $x \in \ell(A_{\alpha,t})$ and hence our assertion follows. □

Remark 2. Theorem 2 gives an important inclusion result in the ℓ-ℓ setting that parallels the famous inclusion result that exists between the power series method of summability and its corresponding weighted mean in the c-c setting [1].

Lemma 5. *Suppose $A = \{a_{nk}\}$ is an ℓ-ℓ matrix such that $a_{nk} = 0$ for $k > n, m > s$ (both positive integers); then $\ell(A^s) \subseteq \ell(A^m)$, where the interpretation for A^s and A^m is as given in* [6, p. 28].

Theorem 3. *If $B = A_q$ is an ℓ-ℓ matrix, then B^m is also an ℓ-ℓ matrix (for m a positive integer greater than 1.)*

Proof. Let $x \in \ell$. B is an ℓ-ℓ matrix implies that $x \in \ell(B)$. By Lemma 5, we have $\ell(B) \subseteq \ell(B^m)$ and hence it follows that $x \in \ell(B^m)$. Hence, B^m is an ℓ-ℓ matrix. □

Remark 3. Theorem 3 gives a result that goes parallel to a c-c result given on [6, Thm. 2.4, p. 28].

In Corollary 1, we showed that A_Q is an ℓ-ℓ matrix. Here, a question may be raised as to whether A_Q maps ℓ^p into ℓ for $p > 1$. But this is answered negatively by the following theorem.

Theorem 4. *A_Q does not map ℓ^p into ℓ for $p>1$.*

Proof. Let $A_Q=\{b_{nk}\}$. Note that if $A_{Q,\alpha}$ maps ℓ^p into ℓ, then by [3, Thm. 2], we must have

$$\lim_k \sum_{n=1}^{\infty}|b_{nk}|=0. \tag{4.16}$$

Let

$$R_n=\sum_{k=1}^{n}Q_k, \tag{4.17}$$

then it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}b_{nk}&=\binom{k+\alpha+1}{k}\sum_{n=k}^{\infty}\frac{1}{R_n}=\binom{k+\alpha+1}{k}\sum_{n=k}^{\infty}\frac{1}{\binom{n+\alpha+2}{n}}\\&\geq M_1k^{\alpha+1}\sum_{n=k}^{\infty}\frac{1}{n^{\alpha+2}}\quad\text{for some } M_1>0\\&\geq M_1M_2k^{\alpha+1}\int_k^{\infty}\frac{dx}{x^{\alpha+2}}\quad\text{for some } M_2>0\\&=\frac{M_1M_2}{\alpha+1}>0.\end{aligned} \tag{4.18}$$

Thus, it follows that

$$\lim_k \sum_{n=1}^{\infty}|b_{nk}|>0, \tag{4.19}$$

and hence A_Q does not map ℓ^p into ℓ for $p>1$ by [3, Thm. 2]. □

Our next theorem has the form of an extension mapping theorem. It indicates that a mapping of A_q from G or G_w into ℓ can be extended to a mapping of ℓ into ℓ.

Theorem 5. *The following statements are equivalent:*
(1) *A_q is an ℓ-ℓ matrix;*
(2) *A_q is a G-ℓ matrix;*
(3) *A_q is a G_w-ℓ matrix.*

Proof. Since G is a subset of ℓ and G_w a subset of G, (1)⇒(2)⇒(3) follow easily. The assertion that (3)⇒(1) follows by [7, Thm. 1.1] and Theorem 1. □

Corollary 4. (1) *A_Q is a G-ℓ matrix.*
(2) *A_Q a G_w-ℓ matrix.*

Proof. Since A_Q is an ℓ-ℓ matrix by Corollary 1, the assertion follows by Theorem 5. □

Corollary 5. (1) *If A_q is a G-G matrix, then A_q is an ℓ-ℓ matrix.*
(2) *If A_q is a G_w-G_w matrix, then A_q is an ℓ-ℓ matrix.*

Proof. The assertion follows easily by Theorem 5. □

Theorem 6. *A_q is a G-G matrix if and only if $1/Q \in G$.*

Proof. If A_q is a G-G matrix, then the first column of A_q is must in G. This gives $1/Q \in G$ since $a_{n,0} = q_0/Q_n$. Conversely, suppose $1/Q \in G$. Then $1/Q_n \le M_1 r^n$ for $M_1 > 0$ and $r \in (0,1)$. Now let $u \in G$, say $|u_k| \le M_2 t^k$ for some $M_2 > 0$ and $t \in (0,1)$. Let Y be the A_q-transform of the sequence u. Then we have

$$|Y_n| \le M_1 M_2 r^n \sum_{k=0}^{n} \binom{k+\alpha}{k} t^k < M_1 M_2 r^n (1-t)^{-(\alpha+1)} < M_3 r^n \quad \text{for some} M_3 > 0. \tag{4.20}$$

Therefore, $Y \in G$ and hence it follows that A_q is a G-G matrix. □

Theorem 7. *A_q is a G_w-G_w matrix if and only if $1/Q \in G_w$.*

Proof. The proof follows easily using the same steps as in the proof of Theorem 6 by replacing G with G_w. □

Lemma 6. *If the Abel-type matrix $A_{\alpha,t}$* [5] *is a G-G matrix, then $A_{\alpha+1,t}$ is also a G-G matrix.*

Proof. By [5, Thm. 7], $A_{\alpha,t}$ is G-G implies that $(1-t)^{\alpha+1} \in G$. But $(1-t)^{\alpha+1} \in G$ yields $(1-t)^{\alpha+2} \in G$, and hence by [5, Thm. 7], it follows that $A_{\alpha+1,t}$ is a G-G matrix. □

Lemma 7. *If the Abel-type matrix $A_{\alpha,t}$* [5] *is a G_w-G_w matrix, then $A_{\alpha+1,t}$ is also a G_w-G_w matrix.*

Proof. The assertion easily follows by replacing G with G_w in the proof of Lemma 6. □

Theorem 8. *If $A_{\alpha,t}$* [5] *and A_q are G-G matrices, then the $G(A_{\alpha,t})$ contains $G(A_q)$.*

Proof. The proof easily follows using the same techniques as in the proof of Theorem 3 by replacing ℓ with G and applying Lemma 6. □

Theorem 9. *If $A_{\alpha,t}$* [3] *and A_q are G_w-G_w matrices, then $G_w(A_{\alpha,t})$ contains $G_w(A_q)$.*

Proof. The proof easily follows using the same techniques as in the proof of Theorem 3 by replacing ℓ with G_w and applying Lemma 7. □

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