Internat. J. Math. & Math. Sci.
Vol. 24, No. 3 (2000) 163–172
S0161171200003331

COMPLETION OF A CAUCHY SPACE WITHOUT THE T_2-RESTRICTION ON THE SPACE

NANDITA RATH

(Received 5 April 1999 and in revised form 27 April 1999)

ABSTRACT. A completion of a Cauchy space is obtained without the T_2 restriction on the space. This completion enjoys the universal property as well. The class of all Cauchy spaces with a special class of morphisms called s-maps form a subcategory CHY′ of CHY. A completion functor is defined for this subcategory. The completion subcategory of CHY′ turns out to be a bireflective subcategory of CHY′. This theory is applied to obtain a characterization of Cauchy spaces which allow regular completion.

Keywords and phrases. Cauchy space, Cauchy map, s-map, stable completion, completion in standard form, regular completion.

2000 Mathematics Subject Classification. Primary 54D35, 54A20, 54B30.

1. Introduction. The completion of Cauchy spaces is already well known and familiar to most of us. In fact, since Keller [5] introduced the axiomatic definition of Cauchy spaces a very rich and extensive completion theory has been developed for Cauchy spaces during the last three decades [2, 3, 7, 10, 12]. It seems that Cauchy space rather than uniform convergence space is a natural generalization of completion of uniform space. But the completion theory developed so far is not so general in nature as the Cauchy spaces considered are T_2 Cauchy spaces [3, 6, 7, 8]. So the natural question arises, "whether a satisfactory completion theory can be developed for a Cauchy space without the T_2-restriction on the space." This question has been partially answered in this paper. A completion functor has been constructed for a subcategory of CHY. This new subcategory is constructed by taking all the Cauchy spaces as objects and morphisms as certain special type of Cauchy maps which we call s-maps.

2. Preliminaries. The following are some basic definitions and notations which we will use throughout the paper. A *filter* on a set X is a nonempty collection of nonempty subsets of X which is closed under finite intersection and formation of supersets. Let $\mathbf{F}(X)$ be the set of filters on X. If $\mathfrak{F},\mathfrak{G} \in \mathbf{F}(X)$, then $\mathfrak{G} \geq \mathfrak{F}$ if and only if for each $F \in \mathfrak{F}$, $\exists G \in \mathfrak{G}$ such that $G \subseteq F$. This defines a partial order relation on $\mathbf{F}(X)$. If $\mathfrak{B}$ is a *base* [11] of the filter $\mathfrak{F}$, then we write $\mathfrak{F} = [\mathfrak{B}]$ and $\mathfrak{F}$ is said to be *generated* by $\mathfrak{B}$. A filter $\mathfrak{F} \in \mathbf{F}(X)$ is said to have a *trace* on $A \subseteq X$, if $F \cap A \neq \phi$, the empty set, $\forall F \in \mathfrak{F}$. In this case, $\mathfrak{F}_A = [\{F \cap A \mid F \in \mathfrak{F}\}]$ is called the *trace* of $\mathfrak{F}$ on A. $\dot{x} = [\{x\}]$ is the filter generated by the singleton set $\{x\}$ and $\mathfrak{F} \cap \mathfrak{G} = [\{F \cup G \mid F \in \mathfrak{F},\ G \in \mathfrak{G}\}]$. If $F \cap G \neq \phi$, $\forall F \in \mathfrak{F}$ and $\forall G \in \mathfrak{G}$, then $\mathfrak{F} \vee \mathfrak{G}$ is the filter $[\{F \cap G \mid F \in \mathfrak{F},\ G \in \mathfrak{G}\}]$. If $\exists F \in \mathfrak{F}$ and $G \in \mathfrak{G}$ such that $F \cap G = \phi$, then we say that $\mathfrak{F} \vee \mathfrak{G}$ *fails to exist.*

In 1968, Keller [5] introduced the following axiomatic definition of Cauchy spaces.

Definition 2.1. A *Cauchy structure* on X is a subset $C \subseteq \mathbf{F}(X)$ satisfying the following conditions:

(c_1) $\dot{x} \in C$, $\forall x \in X$.

(c_2) $\mathfrak{F} \in C$, $\mathfrak{G} \geq \mathfrak{F}$ imply that $\mathfrak{G} \in C$.

(c_3) $\mathfrak{F}, \mathfrak{G} \in C$ and $\mathfrak{F} \vee \mathfrak{G}$ exists imply that $\mathfrak{F} \cap \mathfrak{G} \in C$.

The pair (X, C) is called a *Cauchy space*. If C and D are two Cauchy structures on X and $C \subseteq D$, then C is *finer* than D, written $C \geq D$. For a Cauchy structure C on X we define an equivalence relation "$\sim$" by $\mathfrak{F} \sim \mathfrak{G}$ if and only if $\mathfrak{F} \cap \mathfrak{G} \in C$. Let $[\mathfrak{F}]$ denote the equivalence class determined by $\mathfrak{F}$. Also, there is a convergence structure q_c [7] associated with C in a natural way:

$$\mathfrak{F} \xrightarrow{q_c} x \quad \text{if and only if } \mathfrak{F} \sim \dot{x}. \tag{2.1}$$

A Cauchy space (X, C) is said to be

- T_2 or *Hausdorff* if and only if $x = y$, whenever $\dot{x} \sim \dot{y}$.
- *Regular* if and only if $\text{cl}_{q_c} \mathfrak{F} \in C$, whenever $\mathfrak{F} \in C$, where "cl_{q_c}" is the *closure operator* for q_c.
- *Complete* if and only if each $\mathfrak{F} \in C$ q_c converges.

Note that most of the literature on Cauchy spaces including the completion theory [3, 6, 7, 8, 10] deals exclusively with T_2 Cauchy spaces. The underlying reason for this is the existence of the unique limits in T_2 spaces, which guarantees pleasant consequences. But the T_2 condition is quite restrictive, so our object is to construct a completion theory for Cauchy spaces in general.

A map $f : (X, C) \to (Y, K)$ is called a *Cauchy map*, if $f(\mathfrak{F}) \in K$ whenever $\mathfrak{F} \in C$. The map f is a *homeomorphism*, if f is bijective and f, f^{-1} are both Cauchy maps. If $A \subseteq X$ then $C_A = \{\mathfrak{G} \in \mathbf{F}(A) \mid \exists \mathfrak{F} \in C \text{ such that } \mathfrak{G} \geq \mathfrak{F}_A\}$ is a Cauchy structure on A, called a *subspace structure* on A. The pair (A, C_A) is a *subspace* of (X, C). The mapping $f : (X, C) \to (Y, K)$ is an *embedding* of (X, C) into (Y, K), if $f : (X, C) \to (f(X), K_{f(X)})$ is a homeomorphism, where $K_{f(X)}$ is the subspace structure on $f(X)$.

3. Cauchy space completion. Note that throughout this section (X, C) denotes a Cauchy space (not necessarily T_2), unless otherwise stated. A *completion* of a Cauchy space (X, C) is a pair $((Y, K), \varphi)$ consisting of a complete Cauchy space (Y, K) and an embedding $\varphi : (X, C) \to (Y, K)$ satisfying the condition $\text{cl}_{q_k} \varphi(X) = Y$.

We construct a completion space $((\tilde{X}, \tilde{C}), j)$ of the Cauchy space (X, C) in the following way:

$$\begin{aligned}
&\tilde{X} = \{\dot{x} \mid x \in X\} \cup \{[\mathfrak{F}] \mid \mathfrak{F} \in C \text{ is } q_c \text{ non-convergent}\}, \\
&j : X \longrightarrow \tilde{X} \text{ is defined by } j(x) = \dot{x}, \\
&\tilde{C} = \{\mathfrak{A} \in \mathbf{F}(\tilde{X}) \mid \exists \mathfrak{F} \in C \; q_c \text{ convergent such that } \mathfrak{A} \geq j(\mathfrak{F}) \text{ or } \exists\, \mathfrak{F} \in C \\
&\qquad q_c \text{ non-convergent such that } \mathfrak{A} \geq j(\mathfrak{F}) \cap [\dot{\mathfrak{F}}]\}.
\end{aligned} \tag{3.1}$$

Definition 3.1. A completion $((Y, K), \varphi)$ of a Cauchy space (X, C) is said to be in *standard form*, if $Y = \tilde{X}$, $\varphi = j$, and $j(\mathfrak{F}) \xrightarrow{q_k} [\mathfrak{F}]$ for each q_c non-convergent filter $\mathfrak{F}$ in C.

This notion of completion was given by Reed [10] for T_2 Cauchy spaces. Also, there is an equivalence relation defined between T_2 completions. For a T_2 Cauchy space (Z,D), a T_2 completion $((Y_1,K_1),\varphi_1) \geq ((Y_2,K_2),\varphi_2)$, if there exists a Cauchy map $h:(Y_1,K_1) \to (Y_2,K_2)$ such that $h\circ\varphi_1 = \varphi_2$, and the two completions are equivalent, if each of these two completions is greater than or equal to the other. Note that in case of equivalence, h is a unique Cauchy homeomorphism.

The same definition leads to an equivalence relation between the non-T_2 completions of a non-T_2 Cauchy space (X,C), but it is not a categorical equivalence in the sense of Preuss [9, Proposition 0.2.4], because, as shown in the following examples, h is not necessarily a unique Cauchy homeomorphism.

Example 3.2. Let (R,C) be the set of real numbers with the usual Cauchy structure [1], let $Y = R\cup\{a\}$, where $a \notin R$, and $Z = Y\cup\{b\}$, where $b\notin Y$. Let $\mathfrak{H}$ be an ultrafilter on R finer than the filter generated by the sequence of natural numbers. If $\mathfrak{F}\in \mathbf{F}(R)$, let $\mathfrak{F}'$ and $\mathfrak{F}''$ denote the filters generated by $\mathfrak{F}$ on Y and Z, respectively. Let $C' = C\cup\{\mathfrak{H}\}$ be a Cauchy structure on R. Note that $q_{c'}$ is the usual topology on R and $\mathfrak{H}$ is not $q_{c'}$ convergent.

Let D be the Cauchy structure on Y generated by $\{\mathfrak{F}' \mid \mathfrak{F}\in C\}\cup\{\mathfrak{H}\cap\dot{a}\}$ and K be the Cauchy structure on Z generated by $\{\mathfrak{F}'' \mid \mathfrak{F}\in C\}\cup\{\mathfrak{H}\cap\dot{a}\cap\dot{b}\}$. Observe that (Y,D) and (Z,K) are equivalent completions of (R,C'), if Reed's definition of equivalence described above is generated to non-T_2 completions. But they are not Cauchy homeomorphic, since (Y,D) is T_2 while (Z,K) is not. Furthermore, the functions establishing this equivalence are not unique, since $h_1,h_2:(Y,D)\to(Z,K)$ will both work, where

$$h_1(y) = \begin{cases} y, & y\in R,\\ a, & y\notin R,\end{cases}\qquad h_2(y) = \begin{cases} y, & y\in R,\\ b, & y\notin R.\end{cases} \tag{3.2}$$

Example 3.3. Let (R,C) and Y be as in Example 3.2. Let D' be the Cauchy structure on Y generated by $\{\mathfrak{F}:\mathfrak{F}\in C\}\cup\{\dot{0}\cap\dot{a}\}$. Since (R,C) is complete, it is trivially a completion of itself. Furthermore, (Y,D') is another completion of (R,C) equivalent to (R,C) in the sense described in the preceding example, but not Cauchy homeomorphic to (R,C).

Both Examples 3.2 and 3.3 provide the motivation for introducing the following notion of *stable* completion to ensure appropriate categorical behavior for our completion for the whole class of Cauchy spaces.

Definition 3.4. A completion $((Y,K),\varphi)$ of (X,C) is said to be *stable* if whenever $z\in Y\setminus\varphi(x)$ and $\varphi(\mathfrak{F}) \xrightarrow{q_k} z$ for some $\mathfrak{F}\in C$, it follows that z is the unique limit of $\varphi(\mathfrak{F})$ in Y.

Definition 3.5. A stable completion $\kappa_1 = ((Y_1,K_1),\varphi_1)$ of a Cauchy space (X,C) is said to be *finer* than another stable completion $\kappa_2 = ((Y_2,K_2),\varphi_2)$, if there exists a

Cauchy map $h:(Y_1,K_1)\to(Y_2,K_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} (X,C) & \xrightarrow{\varphi_2} & (Y_2,K_2) \\ {\scriptstyle\varphi_1}\downarrow & \nearrow_{h} & \\ (Y_1,K_1). & & \end{array} \tag{3.3}$$

If κ_1 is finer than κ_2, we write $\kappa_1 \geq \kappa_2$ and $\kappa_2 \geq \kappa_1$. If $\kappa_1 \geq \kappa_2$, then we say that the two completions κ_1 and κ_2 are *equivalent*.

Observe that by taking stable completions in defining the equivalence relation, we get the unique limit of filters which converge to points in $Y_1\backslash\varphi_1(X)$ and $Y_2\backslash\varphi_2(X)$. This ensures that h is a unique Cauchy homeomorphism.

Proposition 3.6. *$((\tilde{X},\tilde{C}),j)$ is the finest stable completion of (X,C) in standard form.*

Proof. It is easy to see that j is injective and $\tilde{C}$ satisfies (c_1) and (c_2) of Definition 2.1. To prove (c_3) let $\mathfrak{A},\mathfrak{B}\in\tilde{C}$ and $\mathfrak{A}\vee\mathfrak{B}$ exist. We consider the following three cases to show that $\mathfrak{A}\cap\mathfrak{B}\in\tilde{C}$

(1) $\mathfrak{A}\geq j(\mathfrak{f})$ and $\mathfrak{B}\geq j(\mathfrak{g})$, where $\mathfrak{f},\mathfrak{g}\in C$ are both q_c convergent,

(2) $\mathfrak{A}\geq j(\mathfrak{f})\cap[\dot{\mathfrak{f}}]$, where $\mathfrak{f}$ is q_c non-convergent, and $\mathfrak{B}\geq j(\mathfrak{g})$, where $\mathfrak{g}$ is q_c convergent,

(3) $\mathfrak{A}\geq j(\mathfrak{f})\cap[\dot{\mathfrak{f}}]$ and $\mathfrak{B}\geq j(\mathfrak{g})\cap[\dot{\mathfrak{g}}]$, where both $\mathfrak{f}$ and $\mathfrak{g}$ are q_c non-convergent.

Proof of (1) is easy and (3) can be proved the same way as (1) and (2), so we prove only (2).

In case (2), $\mathfrak{A}\vee\mathfrak{B}$ exists implies $j(\mathfrak{f})\vee j(\mathfrak{g})$ exists or $j(\mathfrak{g})\vee[\dot{\mathfrak{f}}]$ exists. Since the latter is an impossibility, $j(\mathfrak{f})\vee j(\mathfrak{g})$ exists. This implies that the q_c non-convergent filter $\mathfrak{f}\cap\mathfrak{g}\in C$ and since $\mathfrak{A}\cap\mathfrak{B}\geq j(\mathfrak{f}\cap\mathfrak{g})\cap[\mathfrak{f}\dot{\cap}\mathfrak{g}]$, $\mathfrak{A}\cap\mathfrak{B}\in\tilde{C}$. We conclude that $\tilde{C}$ is a Cauchy structure on $\tilde{X}$.

It is routine to show that $((\tilde{X},\tilde{C}),j)$ is a completion of (X,C) in standard form. To prove that this is also a stable completion, let $\mathfrak{f}\in C$ be q_c non-convergent. If there exists $[\mathfrak{g}]\neq[\mathfrak{f}]\in\tilde{X}\backslash j(X)$ such that $j(\mathfrak{f})\xrightarrow{q_{\tilde{c}}}[\mathfrak{g}]$, then $[\dot{\mathfrak{f}}]\cap[\dot{\mathfrak{g}}]\in\tilde{C}$. This implies that there exists a q_c non-convergent $\mathfrak{h}\in C$ such that $[\dot{\mathfrak{f}}]\cap[\dot{\mathfrak{g}}]\geq j(\mathfrak{h})\cap[\dot{\mathfrak{h}}]$. So $[\mathfrak{f}]=[\mathfrak{h}]=[\mathfrak{g}]$, which lead to a contradiction. Next, let $\mathfrak{f}\in C$ be q_c convergent. If $j(\mathfrak{f})\xrightarrow{q_{\tilde{c}}}[\mathfrak{g}]$, where $[\mathfrak{g}]\in\tilde{X}\backslash j(X)$, then $j(\mathfrak{f})\cap[\dot{\mathfrak{g}}]\in\tilde{C}$. If $j(\mathfrak{f})\cap[\dot{\mathfrak{g}}]\geq j(\mathfrak{C})$, for some convergent filter $\mathfrak{C}\in C$, then $[\mathfrak{g}]=\dot{x}$, for some $x\in X$ which leads to a contradiction. If, on the other hand, $j(\mathfrak{f})\cap[\dot{\mathfrak{g}}]\geq j(\mathfrak{L})\cap[\dot{\mathfrak{L}}]$, where $\mathfrak{L}$ is q_c non-convergent, then $\mathfrak{f}\geq\mathfrak{L}$. But since this implies that $\mathfrak{L}q_c$ converges, we have a contradiction. This proves that $(\tilde{X},\tilde{C})$ is a stable completion. Also it can be easily shown that it is the finest stable completion in standard form.

This completes the proof of Proposition 3.6. □

We call $((\tilde{X},\tilde{C}),j)$ the *Wyler completion* of (X,C)

Corollary 3.7. *If (X,C) is a T_2 Cauchy space, then $((\tilde{X},\tilde{C}),j)$ is a T_2 completion of (X,C).*

In fact, in this case if we identify $\dot{x}$ with its equivalence class $[\dot{x}]$, then the Wyler completion coincides with $((X^*, C^*), j)$ in [10]. Henceforth, we will refer to the completion $((X^*, C^*), j)$ as the T_2-Wyler completion of a T_2 Cauchy space (X, C).

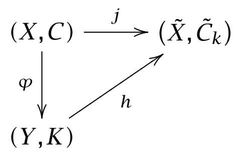
Proposition 3.8. *Any stable completion $((Y,K),\varphi)$ of a Cauchy space (X,C) is equivalent to one in standard form.*

Proof. Define $h : (Y,K) \to (\tilde{X}, \tilde{C})$ as follows

$$h(y) = \begin{cases} [\mathfrak{F}], & \text{if } y \in Y \backslash \varphi(X),\ \varphi(\mathfrak{F}) \xrightarrow{q_k} y, \\ \dot{x}, & \text{if } y = \varphi(x). \end{cases} \tag{3.4}$$

Note that such a non-convergent filter $\mathfrak{F} \in C$ exists, since $((Y,K),\varphi)$ is a completion of (X,C). Since this is also a stable completion, it follows that h is well defined and bijective.

Let $\tilde{C}_k$ be the Cauchy structure on $\tilde{X}$ generated by $\{h(\mathfrak{A}) \mid \mathfrak{A} \in K\}$. Clearly, the diagram



(3.5)

commutes and $((\tilde{X},\tilde{C}_k), j)$ is a completion of (X,C) in standard form.

Also, it can be shown by [9, Propositions 1.2.2.5, 1.2.2.4, and 0.2.7] that h is a Cauchy homeomorphism and therefore, the two completion $((Y,K),\varphi)$ and $((\tilde{X},\tilde{C}_k), j)$ are equivalent. This proves Proposition 3.8. □

In view of Proposition 3.8 all stable completions are equivalent to completions in standard form.

Definition 3.9. A Cauchy map $f : (X,C) \to (Y,D)$ between two Cauchy spaces (X,C) and (Y,D) is said to be an *s-map* if and only if the following condition is satisfied: $\mathfrak{F} \in C$ converges to at most one point in X implies that $f(\mathfrak{F}) \in D$ converges to at most one point in Y.

It is easy to see that the embedding map in any stable completion is an s-map, in particular, the map j in the Wyler completion $((\tilde{X},\tilde{C}), j)$ is an s-map. In fact, any Cauchy map with a T_2 codomain is an s-map. Also, the identity map on any Cauchy space is an s-map and composition of two s-maps is an s-map. So the class of all Cauchy spaces together with the s-maps as morphisms forms a category, which we call CHY$'$. Henceforth, the term *Cauchy category* will be used to denote a category C in which the object are Cauchy spaces and the morphisms are s-maps. In this sense, CHY$'$ and T_2 CHY are Cauchy categories.

Note that every Cauchy map is not necessarily an s-map, for instance, any function from a nontrivial T_2 Cauchy space or an incomplete Cauchy space into an indiscrete Cauchy space containing at least two points is a Cauchy map, but not an s-map. Therefore, CHY$'$ is not a full subcategory of CHY. Furthermore, since there is no s-map from

(R,C) (where R and C are as in Example 3.2) onto a Cauchy space with only two elements, it follows that CHY′ is not closed under the formation of final structure [4] and therefore, it is not a topological category.

But the category CHY′ has other nice properties a few of which we will discuss subsequently. Fric and Kent [3] have shown that any Cauchy map on a T_2 Cauchy space can be uniquely extended to its T_2 Wyler completion. The next proposition shows that the Wyler completion $((\tilde{X},\tilde{C}),j)$ also enjoys this extension property with respect to the s-maps.

Proposition 3.10. *Let $f:(X,C)\to(Y,K)$ be an s-map between two Cauchy spaces (X,C) and (Y,K). Then f has a unique extension $\tilde{f}:(\tilde{X},\tilde{C})\to(\tilde{Y},\tilde{K})$ which is also an s-map and the following diagram commutes:*

$$\begin{array}{ccc} (X,C) & \xrightarrow{\ f\ } & (X,K) \\ {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_Y} \\ (\tilde{X},\tilde{C}) & \xrightarrow[\ \tilde{f}\ ]{} & (\tilde{Y},\tilde{K}). \end{array} \tag{3.6}$$

Proof. $\tilde{f}:(\tilde{X},\tilde{C})\to(\tilde{Y},\tilde{K})$ is defined by

$$\tilde{f}(\dot{x}) = f(\dot{x}), \quad \forall x\in X,$$
$$\tilde{f}([\mathfrak{f}]) = \begin{cases} [f(\mathfrak{f})], & \text{if } f(\mathfrak{f})q_k \text{ non-convergent},\\ \dot{y}, & \text{if } f(\mathfrak{f}) \xrightarrow{q_k} y. \end{cases} \tag{3.7}$$

Since f is an s-map, it follow that $\tilde{f}$ is a well-defined Cauchy map for which the above diagram commutes. To prove that $\tilde{f}$ is an s-map it suffices to show that $\tilde{f}(\mathfrak{A})q_{\tilde{k}}$ converges to only one element in $\tilde{Y}$, whenever $\mathfrak{A}\in\tilde{C}q_{\tilde{c}}$ converges to only one element in $\tilde{X}$. If $\mathfrak{A}\geq j_X(\mathfrak{f})$ for some $\mathfrak{f}\in C$, then $j_X(\mathfrak{f})q_{\tilde{c}}$ converges to only one point in $\tilde{X}$, which in turn implies that $\mathfrak{f}q_c$ converges to only one point in X. Since f and j_Y are s-maps, $\tilde{f}(\mathfrak{A})\geq\tilde{f}\circ j_X(\mathfrak{f}) = j_Y\circ f(\mathfrak{f})q_{\tilde{k}}$ converges to only one element. On the other hand, if $\mathfrak{A}\geq j_X(\mathfrak{g})\cap[\dot{\mathfrak{g}}]$, where $\mathfrak{g}\in C$ is q_c non-convergent, then it suffices to show that $\tilde{f}\circ j_X(\mathfrak{g})\cap\tilde{f}([\dot{\mathfrak{g}}])q_{\tilde{k}}$ converges to only one point. Note that

$$\tilde{f}\circ j_X(\mathfrak{g})\cap\tilde{f}([\dot{\mathfrak{g}}]) = \begin{cases} j_Y\circ f(\mathfrak{g})\cap j_Y(\dot{y}), & \text{if } f(\mathfrak{g}) \xrightarrow{q_k} y,\\ j_Y\circ f(\mathfrak{g})\cap[f(\dot{\mathfrak{f}})], & \text{if } f(\mathfrak{g}) \text{ is } q_k \text{ non-convergent}. \end{cases} \tag{3.8}$$

Since f and j_Y are s-maps, it follows that $j_Y\circ f(\mathfrak{f})$ can converge to at most one point. This shows that $\tilde{f}$ is an s-map.

If $\bar{f}:(\tilde{X},\tilde{C})\to(\tilde{Y},\tilde{K})$ be another s-map which makes the above diagram commute, then obviously $\bar{f}\circ j_X(x) = \tilde{f}\circ j_X(x)$, $\forall x\in X$. So it remains to show that $\bar{f}([\mathfrak{f}]) = \tilde{f}([\mathfrak{f}])$, for each q_c non-convergent filter $\mathfrak{f}\in C$. Since $j_X(\mathfrak{f}) \xrightarrow{q_{\tilde{c}}} [\mathfrak{f}]$, and both $\bar{f},\tilde{f}$ are s-map it follows that $\bar{f}\circ j_X(\mathfrak{f}) \xrightarrow{q_{\tilde{k}}} \bar{f}([\mathfrak{f}])$ and $\tilde{f}\circ j_X(\mathfrak{f}) \xrightarrow{q_{\tilde{k}}} \tilde{f}([\mathfrak{f}])$. Hence $j_Y\circ f(\mathfrak{f}) \xrightarrow{q_{\tilde{k}}} \bar{f}([\mathfrak{f}]),\tilde{f}([F])$. But $\mathfrak{f}\in C$ is q_c non-convergent and f, j_Y are s-maps imply that $j_Y\circ f(\mathfrak{f})$ converges to only one point. Therefore, $\bar{f}([\mathfrak{f}]) = \tilde{f}([\mathfrak{f}])$.

This completes the proof of Proposition 3.10. □

Now we can define a functor on the category CHY′ exactly the same as the T_2 Wyler completion functor defined in [8]. Let $\widetilde{\text{CHY}}'$ be the subcategory of CHY′ consisting of all complete objects in CHY′. We define $\tilde{W} : \text{CHY}' \to \widetilde{\text{CHY}}'$ as follows:

(1) $\tilde{W}(X,C) = (\tilde{X},\tilde{C})$, for all objects (X,C) in CHY′.

(2) $\tilde{W}(f) = \tilde{f}$, for all morphisms f in CHY′, where $\tilde{f}$ is the same as in Proposition 3.10.

Proposition 3.11. *$\tilde{W}$ defined as above is a covariant functor.*

Proof. It follows from Proposition 3.10 that $\tilde{W}(f) = \tilde{f}$ is a morphism in $\widetilde{\text{CHY}}'$, whenever f is a morphism in CHY′. Also, since $\tilde{I}_X(\dot{x}) = I_X(\dot{x}) = \dot{x} = I_{\tilde{X}}(\dot{x})$, $\forall \dot{x} \in \tilde{X}$ and $\tilde{I}_X(\mathfrak{f}) = [I_X(\mathfrak{f})] = [\mathfrak{f}] = I_{\tilde{X}}([\mathfrak{f}])$, $\forall [\mathfrak{f}] \in \tilde{X}$. This shows that $\tilde{W}(I_X) = I_{\tilde{W}}(X)$.

Next we show that $\tilde{W}$ preserves the composition of s-maps. Let $f : (X,C) \to (Y,K)$ and $g : (Y,K) \to (Z,S)$. It is easy to see that $\tilde{W}(f \circ g)(j_X(X)) = (\tilde{f} \circ \tilde{g})(j_X(X)) = (\tilde{W}(f) \circ \tilde{W}(g))(j_X(X))$. We show that $f \tilde{\circ} g([\mathfrak{f}]) = \tilde{f} \circ \tilde{g}([\mathfrak{f}])$, whenever $[\mathfrak{f}] \in \tilde{X} \backslash j_X(X)$. Note that

$$f \tilde{\circ} g([\mathfrak{f}]) = \begin{cases} [f \circ g(\mathfrak{f})], & \text{if } f \circ g(\mathfrak{f}) \text{ is } q_s \text{ non-convergent}, \\ \dot{z}, & \text{if } f \circ g(\mathfrak{f}) \xrightarrow{q_s} z. \end{cases} \tag{3.9}$$

Since $\mathfrak{f}$ is q_c non-convergent and g is an s-map, $g(\mathfrak{f}) q_k$ converges to at most one point in Y. So

$$\tilde{g}([\mathfrak{f}]) = \begin{cases} [g(\mathfrak{f})], & \text{if } g(\mathfrak{f}) \text{ is } q_k \text{ non-convergent}, \\ \dot{y}, & \text{if } g(\mathfrak{f}) \xrightarrow{q_k} y. \end{cases} \tag{3.10}$$

Therefore, it follows that

$$\tilde{f} \circ \tilde{g}([\mathfrak{f}]) = \begin{cases} \tilde{f}[g(\mathfrak{f})], & \text{if } g(\mathfrak{f}) \text{ is } q_k \text{ non-convergent}, \\ \tilde{f}(\dot{y}), & \text{if } g(\mathfrak{f}) \xrightarrow{q_k} y. \end{cases} \tag{3.11}$$

If $g(\mathfrak{f}) \xrightarrow{q_k} y$, then $f \circ g(\mathfrak{f}) \xrightarrow{q_s} f(y)$. Since $\mathfrak{f}$ is q_c non-convergent and $f \circ g$ is an s-map, $f(y) = z$, which shows that $f \tilde{\circ} g(\mathfrak{f}) = \tilde{f} \circ \tilde{g}(\mathfrak{f})$ whenever $f \circ g(\mathfrak{f})$ is q_s convergent. On the other hand, if $g(\mathfrak{f})$ is q_k non-convergent, then $f \circ g(\mathfrak{f})$ converges to at most one point. Observe that

$$\tilde{f}([g(\mathfrak{f})]) = \begin{cases} [f \circ g(\mathfrak{f})], & \text{if } f \circ g(\mathfrak{f}) \text{ is } q_s \text{ non-convergent}, \\ \dot{t}, & \text{if } f \circ g(\mathfrak{f}) \xrightarrow{q_s} t. \end{cases} \tag{3.12}$$

Since $f \circ g(\mathfrak{f})$ converges to at most one point, $t = z$. Therefore, $\tilde{f} \circ \tilde{g}([\mathfrak{f}]) = f \tilde{\circ} g([\mathfrak{f}])$, i.e., $\tilde{W}(f \circ g) = \tilde{f} \circ \tilde{g}$. This proves Proposition 3.11. □

The following lemma describes a condition for the epimorphisms in the category CHY′.

Lemma 3.12. *Let (X,C) and (Y,K) be in* CHY′. *A morphism $f:(X,C)\to(Y,K)$ is an epimorphism if there exists a q_c non-convergent filter $\mathfrak{f}\in C$ such that $f(\mathfrak{f})\xrightarrow{q_k} y$, whenever $y\in Y\backslash f(X)$.*

Proof. Let $\alpha:(Y,K)\to(Z,S)$ and $\beta:(Y,K)\to(T,U)$ be two s-maps such that $\alpha\circ f=\beta\circ f$. For each $y\in Y\backslash f(X)$, there exists a q_c non-convergent filter $\mathfrak{f}\in C$ such that $f(\mathfrak{f})\xrightarrow{q_k} y$. α and β are s-maps imply that $\alpha\circ f(\mathfrak{f})\xrightarrow{q_s}\alpha(y)$ only and $\beta\circ f(\mathfrak{f})\xrightarrow{q_u}\beta(y)$ only. But, since $\alpha\circ f(\mathfrak{f})=\beta\circ f(\mathfrak{f})$ and $\alpha\circ f$ is an s-map, it follows that $\alpha(y)=\beta(y)$. Therefore, $\alpha=\beta$, which implies that f is an epimorphism. □

Note that the embedding map j in the Wyler completion $((\tilde{X},\tilde{C}),j)$ of a Cauchy space (X,C) is an epimorphism.

Proposition 3.13. *In the Cauchy category* CHY′ *Wyler completion is the finest completion of a Cauchy space.*

Proof. Let (X,C) be a Cauchy space and $\kappa=((Y,K),\psi)$ be a completion of (X,C) in the category CHY′, i.e., ψ and ψ^{-1} are s-maps. We show that the Wyler completion is finer than κ.

Define a map $h:(\tilde{X},\tilde{C})\to(Y,K)$ as $h(\dot{x})=\psi(x)$, $\forall x\in X$ and for each q_c non-convergent filter $\mathfrak{f}\in C$, $h([\mathfrak{f}])=y$, where $\psi(\mathfrak{f})\xrightarrow{q_k} y$. Since ψ is an s-map, h is well defined. Also the following diagram commutes:

$$\begin{array}{ccc} (X,C) & \xrightarrow{\ \psi\ } & (Y,K) \\ {\scriptstyle j}\big\downarrow & \nearrow{\scriptstyle h} & \\ (\tilde{X},\tilde{C}). & & \end{array} \tag{3.13}$$

Next we show that h is an s-map. It is routine to show that h is a Cauchy map. Let $\mathfrak{A}\in\tilde{C}$ converge to only one point. If $\mathfrak{A}\geq j(\mathfrak{f})$, then $\mathfrak{f}$ q_c converges to only one point. Since ψ is an s-map, $\psi(\mathfrak{f})$ q_k converges to only one point. Therefore, $h(\mathfrak{A})$ converges to only one point. If $\mathfrak{A}\geq j(\mathfrak{g})\cap[\dot{\mathfrak{g}}]$, where $\mathfrak{g}$ is q_c non-convergent, then $\psi(\mathfrak{g})$ converges to at most one point, say y. But, since $h([\mathfrak{g}])=y$, $h(\mathfrak{A})\geq h\circ j(\mathfrak{g})\cap h([\dot{\mathfrak{g}}])$ converges only to y. This shows that h is an s-map. Therefore, Wyler completion is finer than k in CHY′. This completes the proof of Proposition 3.13. □

From Lemma 3.12 and Proposition 3.13, we obtain the following property of the subcategory $\widetilde{\text{CHY}}'$.

Proposition 3.14. $\widetilde{\text{CHY}}'$ *is an epireflective subcategory of* CHY′.

For a Cauchy category $\mathfrak{A}$, let $\tilde{\mathfrak{A}}$ denote the full subcategory of all complete objects in $\mathfrak{A}$. A Cauchy category $\mathfrak{A}$ is said to be a *Cauchy completion category*, if there is a reflector [9] $R:\mathfrak{A}\to\tilde{\mathfrak{A}}$ such that for each object (X,C) in $\mathfrak{A}$, $(R(X,C),\varphi x)$ is a completion of (X,C), where the embedding map φ_X is a morphism in $\mathfrak{A}$. The reflector R is unique up to equivalence and is called a *completion functor*. In view of Proposition 3.14, CHY′ is a Cauchy completion category and $\tilde{W}$ is a completion functor, called the *Wyler* *completion functor*. Any subcategory of CHY$'$ which admits a completion functor is called a *completion subcategory* of CHY$'$. Note that by taking s-maps as morphisms a completion functor could be defined on the Cauchy space unlike the completion functors in [3, 8], which were restricted to T_2CHY and its subcategories.

Kent and Richardson [8], have constructed completion for T_3 Cauchy space. An attempt has been made to construct a regular completion of a Cauchy space. In fact, in the next proposition we show that the following condition is a necessary and sufficient condition for a Cauchy space (X,C) to have a regular completion:

$(*)$ $\mathfrak{F} \notin C$ implies that there exists a complete regular Cauchy space (Y,K) and an s-map $f:(X,C) \to (Y,K)$ such that $f(\mathfrak{F}) \notin K$.

Note that any complete regular Cauchy space preserves this property.

Lemma 3.15. *A Cauchy space (X,C) has a regular, stable completion if and only if $((\tilde{X},r\tilde{C}),j)$ is a regular completion of (X,C).*

Proof. $(\Rightarrow)$ Let $((Y,D),\varphi)$ be a regular, stable completion of (X,C). Then by Proposition 3.8, $((Y,D),\varphi)$ is equivalent to a regular, stable completion $((\tilde{X},D'),j)$ in standard form. By Proposition 3.6, $D' \le \tilde{C}$ and hence $D' \le r\tilde{C}$. Since $(\tilde{X},D')$ and $(\tilde{X},\tilde{C})$ are both completion of (X,C), so is $((\tilde{X},r\tilde{C}),j)$.

$(\Leftarrow)$ Let $((\tilde{X},r\tilde{C}),j)$ be a regular completion of (X,C). We show that this is also a stable completion. Let $z = [\mathfrak{g}] \in \tilde{X}\setminus j(X)$ and assume $j(\mathfrak{F}) \xrightarrow{q_{r\tilde{c}}} z$, where $\mathfrak{F} \in C$. If $\mathfrak{F}$ q_c converges, then $j(\mathfrak{F}\cap\mathfrak{g}) = j(\mathfrak{F})\cap j(\mathfrak{g}) \in r\tilde{C}$, which implies that $\mathfrak{F}\cap\mathfrak{g} \in C$, a contradiction since $\mathfrak{g}$ is not q_c convergent. If $\mathfrak{F}$ is not q_c convergent and $[\mathfrak{F}] \ne [\mathfrak{g}]$ then $j(\mathfrak{F})q_{r\tilde{c}}$ converges to $[\mathfrak{F}]$ and $[\mathfrak{g}]$, whence $j(\mathfrak{F}\cap\mathfrak{g}) \in r\tilde{C}$. This implies that $\mathfrak{F}\cap\mathfrak{g} \in C$, which is again a contradiction. Thus, $[\mathfrak{F}] = [\mathfrak{g}]$ is the unique limit of $j(\mathfrak{F})$ in $(\tilde{X},r\tilde{C})$, and the completion $((\tilde{X},r\tilde{C}),j)$ is stable. □

Proposition 3.16. *A Cauchy space (X,C) has a regular stable completion if and only if (X,C) satisfies the condition $(*)$.*

Proof. $(\Leftarrow)$ Assume the condition $(*)$. By Lemma 3.15, we need only to show that $((\tilde{X},r\tilde{C}),j)$ is a completion of (X,C). It is routine to show that j is an injective Cauchy map for which $\tilde{X} = \mathrm{cl}_{q_{r\tilde{c}}} j(X)$ and $(\tilde{X},r\tilde{C})$ is complete [6]. So it remains only to show that j^{-1} is a Cauchy map. If not, $\exists\mathfrak{H} \in r\tilde{C}$ such that $j^{-1}\mathfrak{H} \notin C$. By $(*)$ there exists a complete regular Cauchy space (Y,K) and s-map $f:(X,C)\to(Y,K)$ such that $f(j^{-1}\mathfrak{H}) \notin K$. Let $\tilde{f}:(\tilde{X},\tilde{C}) \to (Y,K)$ be the s-extension of f, to the Wyler completion, as in Proposition 3.10. Since (Y,K) is regular, $\tilde{f}:(\tilde{X},r\tilde{C}) \to (Y,K)$ is also a Cauchy map. So $\mathfrak{H} \in r\tilde{C}$ implies that $\tilde{f}(\mathfrak{H}) \in K$. But since $f(j^{-1}(\mathfrak{H})) \ge \tilde{f}(\mathfrak{H})$, $f(j^{-1}(\mathfrak{H})) \in K$, a contradiction. Thus j^{-1} is a Cauchy map, so $((\tilde{X},r\tilde{C}),j)$ is a regular completion of (X,C).

$(\Rightarrow)$ By Lemma 3.15, $((\tilde{X},r\tilde{C}),j)$ is a regular stable completion of (X,C). If $\mathfrak{F} \notin C$, then $j(\mathfrak{F}) \notin r\tilde{C}$, since j^{-1} is a Cauchy map. Also, since this completion is stable, j is an s-map. So (X,C) satisfies $(*)$. This completes the proof of Proposition 3.16. □

Note that if the inverse of an injective Cauchy map is a Cauchy map then it is also an s-map. So j^{-1} in the completion $((\tilde{X},r\tilde{C}),j)$ is an s-map. Hence we have the following corollary.

Corollary 3.17. *If* (X,C) *and* (Y,K) *are two Cauchy spaces satisfying* $(*)$ *and* $f:(X,C)\to(Y,K)$ *is an* s*-map, then* $\tilde{f}:(\tilde{X},r\tilde{C})\to(\tilde{Y},r\tilde{K})$ *is also an* s*-map, where* $\tilde{f}$ *is as in Proposition 3.10.*

Let SCHY′ and RCHY′ be the full subcategories of CHY′ consisting of Cauchy spaces satisfying $(*)$ and regular Cauchy spaces, respectively. Since every complete regular Cauchy space is in SCHY′, $\mathrm{S}\widetilde{\mathrm{CHY}}' = \mathrm{R}\widetilde{\mathrm{CHY}}'$. Define a functor $S:\mathrm{SCHY}'\to\mathrm{S}\widetilde{\mathrm{CHY}}'$ as follows: $S(X,C)=(\tilde{X},r\tilde{C})$ and $S(f)=\tilde{f}$. Proof of the following proposition is now immediate.

Proposition 3.18. *We have that* S *is a completion functor and* SCHY′ *is a completion subcategory of* CHY′*.*

There can be many applications of this theory in the completion of linear Cauchy spaces and Cauchy groups. Fric, Kent and Richardson have studied these spaces with the T_2 restriction on the underlying spaces. However, if we develop this theory without the T_2 restriction, we can generalize several problems in functional analysis. Of course, in those cases we have to look at linear s-maps and stable completions with compatible algebraic structures. The categorical properties like Cartesian closedness of the subcategory CHY′ of CHY also remain to be investigated.

References

[1] N. Bourbaki, *Elements of Mathematics. General Topology. Part 1*, Hermann, Addison Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., Paris, 1966. MR 34#5044a. Zbl 301.54001.

[2] A. Csaszar, *λ-complete filter spaces*, Acta Math. Hungar. **70** (1996), no. 1-2, 75–87. MR 97d:54005. Zbl 855.54038.

[3] R. Fric and D. C. Kent, *Completion functors for Cauchy spaces*, Int. J. Math. Math. Sci. **2** (1979), no. 4, 589–604. MR 80m:54042. Zbl 428.54018.

[4] H. Herrlich and G. E. Strecker, *Category Theory*, An introduction, 2nd ed., Sigma Series in Pure Mathematics, vol. 1, Heldermann Verlag, Berlin, 1979. MR 81e:18001. Zbl 437.18001.

[5] H. H. Keller, *Die Limes-Uniformisierbarkeit der Limesraume.*, Math. Ann. **176** (1968), 334–341. MR 37#874. Zbl 155.50302.

[6] D. C. Kent, *The regularity series of a Cauchy space*, Int. J. Math. Math. Sci. **7** (1984), no. 1, 1–13. MR 85d:54034. Zbl 564.54004.

[7] D. C. Kent and G. D. Richardson, *Regular completions of Cauchy spaces*, Pacific J. Math. **51** (1974), 483–490. MR 52#11811. Zbl 291.54024.

[8] ———, *Cauchy completion categories*, Canad. Math. Bull. **32** (1989), no. 1, 78–84. MR 90a:54007. Zbl 675.54004.

[9] G. Preuss, *Theory of Topological Structures*, Mathematics and its Applications, vol. 39, D. Reidel Publishing Co., Dordrecht, Boston, 1988, An approach to categorical topology. MR 89m:54014. Zbl 649.54001.

[10] E. E. Reed, *Completions of uniform convergence spaces*, Math. Ann. **194** (1971), 83–108. MR 45#1109. Zbl 217.19603.

[11] S. Willard, *General Topology*, Addison Wesley Publishing Co., Reading, Mass. London Don Mills, Ont., 1970. MR 41#9173. Zbl 205.26601.

[12] O. Wyler, *Ein Komplettierungsfunktor fur uniforme Limesraume*, Math. Nachr. **46** (1970), 1–12. MR 44#985. Zbl 207.52603.

Nandita Rath: Department of Mathematics, University of Western Australia, Nedlands, WA 6009, Australia

E-mail address: nandita.rath@mail.atu.edu


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization