Internat. J. Math. & Math. Sci.
Vol. 24, No. 2 (2000) 73–93
S0161171200002593

FIXED POINTS, INTERSECTION THEOREMS, VARIATIONAL INEQUALITIES, AND EQUILIBRIUM THEOREMS

SEHIE PARK

(Received 20 October 1998)

ABSTRACT. From a fixed point theorem for compact acyclic maps defined on admissible convex sets in the sense of Klee, we first deduce collectively fixed point theorems, intersection theorems for sets with convex sections, and quasi-equilibrium theorems. These quasi-equilibrium theorems are applied to give simple and unified proofs of the known variational inequalities of the Hartman-Stampacchia-Browder type. Moreover, from our new fixed point theorem, we deduce new variational inequalities which can be used to obtain fixed point results for convex-valued maps. Finally, various general economic equilibrium theorems are deduced in the forms of the Nash type, the Tarafdar type, and the Yannelis-Prabhakar type. Our results are stated for not-necessarily locally convex topological vector spaces and for abstract economies with arbitrary number of commodities and agents. Our new results extend a lot of known works with much simpler proofs.

Keywords and phrases. Multimap (map), closed map, compact map, upper semicontinuous (u.s.c.), lower semicontinuous (l.s.c.), acyclic map, quasiconcave, quasiconvex, admissible subset of a topological vector space (t.v.s.), fixed point, convex space, polytope, quasi-equilibrium problem, variational inequality, economic equilibrium theorem, abstract economy, equilibrium point, maximal point.

2000 Mathematics Subject Classification. Primary 46A55, 49J40, 49J53, 91B50; Secondary 47H10, 52A07, 54H25, 55M20.

1. Introduction. Recently, we obtained a new fixed point theorem for compact multimaps defined on admissible convex subsets in not-necessarily locally convex topological vector spaces (see [51, 53]). Our theorem is one of the most general results and substantially extends a large number of known theorems, including Kakutani's theorem [26] for Euclidean spaces and Himmelberg's theorem [17] for locally convex topological vector spaces. Because these theorems were so useful in various problems in mathematical sciences including economic and game theories, it seems to be quite natural to generalize known results on applications of theorems of Kakutani and Himmelberg in view of our theorem. In this way, we can treat more general topological vector spaces than locally convex ones.

In the first half of the present paper, from a particular form of our fixed point theorem with the aid of some known selection theorems, we deduce new results on collectively fixed points (see Section 3), intersection theorems for sets with convex sections (see Section 4), and quasi-equilibrium problems (see Section 5). These quasi-equilibrium theorems are applied to give simple and unified proofs of the known variational inequalities of the Hartman-Stampacchia-Browder type (see Section 6).

In the second half, we deduce new variational inequalities (see Section 7) which can

be used to obtain fixed point theorems generalizing a large number of historically well-known extensions of the Brouwer or Kakutani theorems. Finally, various general economic equilibrium theorems are deduced. These are the Nash type (see Section 8), the Tarafdar type (see Section 9), and the Yannelis-Prabhakar type (see Section 10). Our results are stated for not-necessarily locally convex topological vector spaces and for abstract economies with arbitrary number of commodities and agents.

Consequently, our new results extend a lot of known works due to von Neumann [71], Nash [41], Fan [11, 12, 13, 14], Ma [37], Idzik [21, 22], Yannelis and Prabhakar [73], Tarafdar [69], Kim and Tan [30], and others, with much simpler proofs.

2. A fixed point theorem for acyclic maps. A *multimap* or *map* $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values, and $x \in T^-(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(T) = \{(x,y) : x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if for each closed set $B \subset Y$, the set $T^-(B) = \{x \in X : T(x) \cap B = \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if for each open set $B \subset Y$, the set $T^-(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. Note that every u.s.c. map T with closed values is closed.

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. Note that any nonempty convex or star-shaped subset of a topological vector space is contractible, and that any contractible space is acyclic. A map $T : X \multimap Y$ is said to be *acyclic* if it is u.s.c. with acyclic compact values.

Recall that a real-valued function $g : X \to \mathbb{R}$ on a topological space X is *lower* (resp., *upper*) *semicontinuous* (l.s.c.) (resp., u.s.c.) if $\{x \in X : g(x) > r\}$ (resp., $\{x \in X : g(x) < r\}$) is open for each $r \in \mathbb{R}$. If X is a convex set in a vector space, then $g : X \to \mathbb{R}$ is *quasiconcave* (resp., *quasiconvex*) if $\{x \in X : g(x) > r\}$ (resp., $\{x \in X : g(x) < r\}$) is convex for each $r \in \mathbb{R}$.

Berge's theorem (see [5]). *Let X and Y be topological spaces, $f : X \times Y \to \mathbb{R}$ a real function, $F : X \multimap Y$ a multimap, and*

$$\hat{f}(x) = \sup_{y \in F(x)} f(x,y), \quad G(x) = \{y \in F(x) : f(x,y) = \hat{f}(x)\}, \quad \textit{for } x \in X. \tag{2.1}$$

(a) *If f is u.s.c. and F is u.s.c. with compact values, then $\hat{f}$ is u.s.c.*

(b) *If f is l.s.c. and F is l.s.c, then $\hat{f}$ is l.s.c.*

(c) *If f is continuous and F is continuous with compact values, then $\hat{f}$ is continuous and G is u.s.c.*

Throughout this paper, all topological spaces are assumed to be Hausdorff, a t.v.s. is a topological vector space, and co and $\overline{}$ denote the convex hull and closure, respectively.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [31]) provided that, for every compact subset K of X and every neighborhood V of the

origin of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible (see Hukuhara [19] and Nagumo [40]). Other examples of admissible t.v.s. are ℓ^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, the Hardy spaces H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Note also that any locally convex subset of an F-normable t.v.s. is admissible and that every compact convex locally convex subset of a t.v.s. is admissible. For details, see Klee [31], Hadžić [15], Weber [72], and references therein.

The following particular form of our new fixed point theorem in [51, 53] is the basis of our arguments in this paper. We give its proof for completeness.

Theorem 2.1. *Let E be a t.v.s. and let X be an admissible convex subset of E. Then any compact acyclic map $F : X \multimap X$ has a fixed point $x \in X$; that is, $x \in F(x)$.*

Proof. Let υ be a fundamental system of neighborhoods of the origin of E. Since F is closed and compact, it is sufficient to show that for any $V \in \upsilon$, there exists an $x_V \in X$ such that $(x_V + V) \cap F(x_V) \neq \emptyset$.

Since $\overline{F(X)}$ is a compact subset of the admissible subset X, there exists a continuous map $h : \overline{F(X)} \to X$ and a finite dimensional subspace L of E such that $x - h(x) \in V$ for all $x \in \overline{F(X)}$ and $h(\overline{F(X)}) \subset L \cap X$. Let $M := h(\overline{F(X)})$. Then M is a compact subset of L and hence $P := \operatorname{co} M$ is a compact convex subset of $L \cap X$. Note that $h : \overline{F(X)} \to P$ and $F|_P : P \to \overline{F(X)}$. Since h and $F|_P$ are acyclic maps, it is well known (see [53]) that their composition $h(F|_P)$ has a fixed point $x_V \in P$; that is, $x_V \in hF(x_V)$ and hence $x_V = h(y)$ for some $y \in F(x_V)$. Since $y - h(y) \in V$, we have $y \in h(y) + V = x_V + V$. Therefore, $(x_V + V) \cap F(x_V) \neq \emptyset$. This completes the proof. □

Note that Theorem 2.1 generalizes many of known fixed point theorems even when E is locally convex, especially, for the case F has convex values (see Kakutani [26], Himmelberg [17], and Park [46, 60]).

Corollary 2.2. *Let X be an admissible compact convex subset of a t.v.s. E and $F : X \multimap E$ a map with closed convex graph. If $F(X) \supset X$, then F has a fixed point.*

Proof. Let $G := F^-|_X : X \multimap X$ be defined by $G(y) = \{x \in X : y \in F(x)\}$ for $y \in X$. Then the graph $\operatorname{Gr}(G)$ of G is the symmetric part of $\operatorname{Gr}(F) \cap (X \times X)$ and, hence, closed and convex. Therefore, G has closed convex values. Now, by Theorem 2.1, G has a fixed point $x \in G(x)$; that is, $x \in F(x)$. This completes the proof. □

Note that if F is single-valued, Corollary 2.2 reduces to Penot [61, Proposition 1.7] whenever E is locally convex, and to Park [50, Theorem 7] whenever E^* separates points of E.

3. Collectively fixed points.

A convex space is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes* (see Lassonde [34]).

Lemma 3.1 (see [55]). *Let X be a topological space, Y a convex space, and $S,T : X \multimap Y$ maps satisfying*

(1) *for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$;*

(2) *for each $y \in Y$, $S^-(y)$ is compactly open in X; or $X = \bigcup\{\operatorname{Int} S^-(y) : y \in Y\}$.*

Then, for any nonempty compact subset K of X, there exists a continuous function $f : K \to Y$ such that

(3) *$f(x) \in T(x)$ for each $x \in K$;*

(4) *$f(K)$ is contained in a polytope of Y;*

(5) *for any compact subset L of X containing K, there exists a continuous extension $\tilde{f} : L \to Y$ of f such that $\tilde{f}(x) \in T(x)$ for each $x \in L$ and $\tilde{f}(L)$ is contained in a polytope of Y.*

Lemma 3.2 (see [18]). *Let X be a paracompact space, Y a convex space, and $A : X \multimap Y$ a map such that $X = \bigcup\{\operatorname{Int} A^-(y) : y \in Y\}$. Then $\operatorname{co} A : X \multimap Y$ has a continuous selection $s : X \to Y$; that is, $s(x) \in \operatorname{co} A(x)$ for all $x \in X$.*

Let $\{X_i\}_{i\in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j\in I} X_j, \qquad X^i = \prod_{j\in I\setminus\{i\}} X_j. \tag{3.1}$$

If $x^i \in X^i$ and $j \in I\setminus\{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its ith coordinate is x_i and, for $j \neq i$, the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x in X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) = \{y_i \in X_i : [x^i, y_i] \in A\}, \qquad A(x_i) = \{y^i \in X^i : [y^i, x_i] \in A\}. \tag{3.2}$$

For a family $\{E_i\}_{i\in I}$ of t.v.s., let $E = \prod_{i\in I} E_i$. Similarly $X = \prod_{i\in I} X_i$ and $K = \prod_{i\in I} K_i$ for subsets X_i and K_i of E_i for $i \in I$.

We begin with the following collectively fixed point theorems.

Theorem 3.3. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and $T_i : X \multimap K_i$ a closed map with convex values for each $i \in I$. If X is admissible in E, then there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

Proof. Define $T : X \multimap K$ by $T(x) = \prod_{i\in I} T_i(x)$ for each $x \in X$. Then $T : X \multimap X$ is a compact closed map with convex values. Since X is admissible, by Theorem 2.1, T has a fixed point $\hat{x} \in K$; that is, $\hat{x} \in T(\hat{x})$ and hence $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$. □

Remarks. (1) If T_i is u.s.c. for each $i \in I$, then T is also u.s.c. This was first shown by Fan [11, Lemma 3].

(2) If each E_i is locally convex, Theorem 3.3 reduces to Idzik [22, Theorem 5] and Kim et al. [28, Theorem 5].

(3) If I is a singleton, Theorem 3.3 reduces to a particular form of Theorem 2.1 for convex-valued maps.

Theorem 3.4. *Let $\{X_i\}_{i\in I}$ and $\{K_i\}_{i\in I}$ be the same as in Theorem 3.3. For each $i \in I$, let $S_i,\ T_i : X \multimap K_i$ be maps satisfying*

(1) *for each $x \in X$, $\operatorname{co} S_i(x) \subset T_i(x)$;*

(2) *$D = \operatorname{co} K \subset \bigcup_{y\in K_i} \operatorname{Int} S_i^-(y)$.*

If D is admissible in E, then there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.

Proof. Since K is compact in E, $D = \operatorname{co} K$ is σ-compact (see Lassonde [35]) and hence Lindelöf. Since D is regular, we know that D is paracompact. Consider $S_i|_D : D \multimap K_i$. Note that $D = \bigcup_{y\in K_i}(\operatorname{Int} S_i^-(y)) \cap D$ by (2) and $\operatorname{Int}_D(S_i|_D)^-(y) = (\operatorname{Int} S_i^-(y)) \cap D$ for $y \in K_i$. Therefore, by Lemma 3.2, $(\operatorname{co} S_i)|_D : D \multimap X_i$ has a continuous selection $s_i : D \to K_i$ such that $s_i(x) \in \operatorname{co} S_i(x) \subset T_i(x)$ for each $x \in D$. Define $s : D \to K$ by $s(x) = \prod_{i\in I} s_i(x)$ for $x \in D$. Since D is an admissible convex subset of E and $s : D \to D$ is a continuous compact map, by Theorem 2.1, s has a fixed point $\hat{x} \in K$; that is, $\hat{x} = s(\hat{x})$ and hence $\hat{x}_i = s_i(\hat{x}) \in T_i(\hat{x})$ for each $i \in I$. This completes the proof. □

Examples. (1) Yannelis and Prabhakar [73, Theorem 3.2]: I is a singleton, E_i is locally convex, $S_i = T_i$, and X_i is paracompact.

(2) Ding, Kim, and Tan [10, Theorem 2]: each E_i is locally convex.

(3) Husain and Tarafdar [20, Theorem 2.2]: each E_i is locally convex.

If all X_i's are compact in Theorem 3.4, we do not need the admissibility of X as follows.

Theorem 3.5. *Let $\{X_i\}_{i\in I}$ be a family of compact convex spaces and, for each $i \in I$, let $S_i,\ T_i : X \multimap X_i$ be maps satisfying*

(1) *$\operatorname{co} S_i(x) \subset T_i(x)$ for each $x \in X$; and*

(2) *$X = \bigcup_{y\in X_i} \operatorname{Int} S_i^-(y)$.*

Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$.

Proof. By Lemma 3.1, for each $i \in I$, T_i has a continuous selection $f_i : X \to K_i$, where K_i is a polytope in X_i. Note that each K_i is a compact convex subset of a finite dimensional space E_i, which is a locally convex t.v.s. Define a map $f : K \to K$ by $f(x) = \prod_{i\in I} f_i(x)$ for $x \in K$. Note that f is continuous and that K is admissible as a convex subset of a locally convex t.v.s. E. Therefore, by Theorem 2.1, we have a fixed point $\hat{x} \in X$ of f; that is, $\hat{x}_i = f_i(\hat{x}) \in T_i(\hat{x})$ for all $i \in I$. This completes the proof. □

Examples. (1) If I is a singleton and $S_i = T_i$, then Theorem 3.5 reduces to the well-known Fan-Browder fixed point theorem (see Park [47]).

(2) For the case I is a singleton, Theorem 3.5 was due to Ben-El-Mechaiekh et al. [4, Theorem 1] and Simons [64, Theorem 4.3]. This was extended by many authors (see Park [47]).

4. Intersection theorems for sets with convex sections. The collectively fixed point theorems in Section 3 can be reformulated to generalize various von Neumann type intersection theorems for sets with convex sections as follows.

Theorem 4.1. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of X such that $A_i(x^i)$ is a nonempty convex subset of K_i for each $x^i \in X^i$ and each $i \in I$. If X is admissible in E, then $\bigcap_{i\in I} A_i \neq \emptyset$.*

Proof. We use Theorem 3.3 with $T_i : X \multimap K_i$ defined by $T_i(x) = A_i(x^i)$ for $x \in X$. Then, for each $x \in X$, we have

$$\begin{aligned}(x,y) \in \operatorname{Gr}(T_i) &\Longleftrightarrow (x_i,x^i) \in X_i \times X^i \text{ and } y \in A_i(x^i) \subset K_i \\ &\Longleftrightarrow (x_i,x^i,y) \in X_i \times (A_i \cap (X^i \times K_i)),\end{aligned} \tag{4.1}$$

which implies that $\operatorname{Gr}(T_i)$ is closed in $X \times K_i$. Hence, each T_i is a closed map with nonempty convex values. Therefore, by Theorem 3.3, there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$. Since $\hat{x}_i \in K_i \subset X_i$, we have $\hat{x} = [\hat{x}^i, \hat{x}_i] \in A_i$ for all $i \in I$. This completes the proof. □

Examples. (1) von Neumann [71]: $I = \{1,2\}$ and E_i are Euclidean.

(2) Fan [11, Theorem 2]: E_i are locally convex and $X_i = K_i$ for all $i \in I$. This result was applied in [11] to obtain a minimax theorem generalizing von Neumann's and Ville's.

(3) Idzik [22, Corollary 1]: each E_i is locally convex.

Note that, as the von Neumann intersection theorem is equivalent to Kakutani's fixed point theorem, Theorems 3.3 and 4.1 are easily seen to be equivalent.

From Theorem 3.4, we have the following equivalent form.

Theorem 4.2. *Let $\{X_i\}_{i\in I}$ and $\{K_i\}_{i\in I}$ be the same as in Theorem 4.1. For each $i \in I$, A_i and B_i are subsets of X satisfying*

(1) *for each $x^i \in X^i$, $\varnothing = \operatorname{co} B_i(x^i) \subset A_i(x^i) \subset K_i$; and*

(2) *for each $y \in K_i$, $B_i(y)$ is open in X^i.*

If $\operatorname{co} K$ is admissible in E, then we have $\bigcap_{i\in I} A_i \neq \varnothing$.

Proof. We apply Theorem 3.4 with $S_i,\ T_i : X \multimap K_i$ given by $S_i(x) = B_i(x^i)$ and $T_i(x) = A_i(x^i)$ for each $x \in X$. Then, for each $i \in I$, we have the following:

(a) for each $x \in X$, we have $\operatorname{co} S_i(x) \subset T_i(x) \subset K_i$;

(b) for each $y \in K_i$,

$$x \in S_i^-(y) \Longleftrightarrow y \in S_i(x) \Longleftrightarrow (x,y) \in \operatorname{Gr}(S_i) \subset X \times K_i \tag{4.2}$$

and, on the other hand,

$$x \in S_i^-(y) \Longleftrightarrow y \in S_i(x) = B_i(x^i) \Longleftrightarrow (x^i,y) \in B_i. \tag{4.3}$$

Hence,

$$S_i^-(y) = \{x = [x^i, x_i] \in X : x^i \in B_i(y),\ x_i \in X_i\} = B_i(y) \times X_i. \tag{4.4}$$

Note that $S_i^-(y)$ is open in $X = X^i \times X_i$.

Therefore, by Theorem 3.4, there exists an $\hat{x} \in K$ such that $\hat{x} \in T\hat{x} = \prod_{i\in I} T_i(\hat{x})$; that is, $\hat{x}_i \in T_i(\hat{x}) = A_i(\hat{x}^i)$ for all $i \in I$. Hence, $\hat{x} = [\hat{x}^i, \hat{x}_i] \in \bigcap_{i\in I} A_i \neq \varnothing$. This completes the proof. □

From Theorem 3.5, we have the following equivalent form.

Theorem 4.3. *Let $\{X_i\}_{i\in I}$ be a family of compact convex spaces and, for each $i \in I$, let A_i and B_i are subsets of X satisfying the following:*

(1) *for each $x^i \in X^i$, $\varnothing = \operatorname{co} B_i(x^i) \subset A_i(x^i)$; and*

(2) *for each $y \in X_i$, $B_i(y)$ is open in X^i.*

Then we have $\bigcap_{i\in I} A_i \neq \varnothing$.

Proof. Apply Theorem 3.5 instead of Theorem 3.4 and follow the proof of Theorem 4.2. □

Examples. (1) Fan [12, Théorème 1]: I is finite and $A_i = B_i$ for all $i \in I$.

(2) Fan [13, Theorem 1′]: $I = \{1,2\}$ and $A_i = B_i$ for all $i \in I$.

From these results, Fan [13] deduced an analytic formulation, fixed point theorems, extension theorems of monotone sets, and extension theorems for invariant vector subspaces.

(3) Ma [37, Theorem 2]: $A_i = B_i$ for all $i \in I$.

(4) Chang [9, Theorem 4.2] first obtained Theorem 4.3 with a different proof. She also obtained a noncompact version of Theorem 4.3 as [9, Theorem 4.3].

5. Quasi equilibrium problems. Theorem 3.3 can be reformulated to the form of a quasi-equilibrium theorem as follows.

Theorem 5.1. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, $S_i : X \multimap K_i$ a closed map, and $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ u.s.c. functions for each $i \in I$. Suppose that for each $i \in I$,*

(1) *$g_i(x) \le f_i(x)$ for each $x \in X$;*

(2) *the function M_i, defined on X by*

$$M_i(x) = \max_{y\in S_i(x)} g_i(x^i, y), \tag{5.1}$$

is l.s.c.; and

(3) *for each $x \in X$, the set*

$$\{y \in S_i(x) : f_i(x^i, y) \ge M_i(x)\}, \tag{5.2}$$

is convex.

If X is admissible in E, then there exists an $\hat{x} \in K$ such that for each $i \in I$,

$$\hat{x}_i \in S_i(\hat{x}), \quad f_i(\hat{x}^i, \hat{x}_i) \ge M_i(\hat{x}). \tag{5.3}$$

Proof. For each $i \in I$, define a map $T_i : X \multimap K_i$ by

$$T_i(x) = \{y \in S_i(x) : f_i(x^i, y) \ge M_i(x)\}, \tag{5.4}$$

for $x \in X$. Note that each $T_i(x)$ is nonempty by (1) since $S_i(x)$ is compact and $g_i(x^i, \cdot)$ is u.s.c. on $S_i(x)$. We show that $\operatorname{Gr}(T_i)$ is closed in $X \times K_i$. In fact, let $(x_\alpha, y_\alpha) \in \operatorname{Gr}(T_i)$ and $(x_\alpha, y_\alpha) \to (x, y)$. Then

$$\begin{aligned} f_i(x^i, y) &\ge \overline{\lim_\alpha} f_i(x^i_\alpha, y_\alpha) \ge \overline{\lim_\alpha} M_i(x_\alpha) \\ &\ge \underline{\lim_\alpha} M_i(x_\alpha) \ge M_i(x) \end{aligned} \tag{5.5}$$

and, since $\mathrm{Gr}(S_i)$ is closed in $X \times K_i$, $y_\alpha \in S_i(x_\alpha)$ implies $y \in S_i(x)$. Hence $(x,y) \in \mathrm{Gr}(T_i)$. Now we apply Theorem 3.3. Then there exists an $\hat{x} \in K$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$; that is, $\hat{x}_i \in S_i(\hat{x})$ and $f_i(\hat{x}^i, \hat{x}_i) \geq M_i(\hat{x})$. This completes the proof. □

Example. Idzik [21, Theorem 7], Marchi and Martínez-Legas [38, Theorem 5.1, Corollary 5.1]: each E_i is locally convex, $f_i = g_i$ is continuous, and S_i is continuous. In this case, (2) follows from Berge's theorem.

If $f_i = g_i = 0$, then Theorem 5.1 reduces to Theorem 3.3.

We have another quasi-equilibrium theorem equivalent to Theorem 2.1 by following the proof of Theorem 5.1.

Theorem 5.2. *Let X be an admissible convex subset of a t.v.s. E, $f : X \times X \to \mathbb{R}$ an u.s.c. function, and $S : X \multimap X$ a compact closed map. Suppose that*

(1) *the function M defined on X by*

$$M(x) = \max_{y \in S(x)} f(x,y) \quad \text{for } x \in X, \tag{5.6}$$

is l.s.c.; and

(2) *for each $x \in X$, the set*

$$\{y \in S(x) : f(x,y) = M(x)\} \tag{5.7}$$

is acyclic.

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in S(\hat{x}), \quad f(\hat{x},\hat{x}) = M(\hat{x}). \tag{5.8}$$

Examples. (1) If $f(x,y) = 0$ for all $x, y \in X$, then Theorem 5.2 reduces to Theorem 2.1. If f and S are continuous, then condition (1) holds by Berge's theorem.

(2) For a locally convex t.v.s. E, particular forms of Theorem 5.2 were obtained by Takahashi [67, Theorem 4] and Im and Kim [24, Theorem 1]. Those authors applied their results to best approximation problems and optimization problems, respectively (see also Park [48] and Park and Chen [56]).

6. Applications to variational inequalities. Theorem 5.2 can be used to give simple proofs of the variational inequalities of the Hartman-Stampacchia-Browder type as follows:

(i) Hartman and Stampacchia [16, Lemma 3.1]: *let K be a compact convex set in $\mathbb{R}^n$ and $B : K \to \mathbb{R}^n$ a continuous map. Then there exists a $u_0 \in K$ such that*

$$\langle B(u_0), v - u_0 \rangle \geq 0 \quad \forall\, v \in K. \tag{6.1}$$

Put $X = K$, $f(x,y) = \langle B(x), -y \rangle$, $S(x) = K$ for $x, y \in K$, and apply Theorem 5.2.

(ii) Browder [6, Theorem 3], [7, Theorem 2]: *let E be a t.v.s. on which its topological dual E^* is equipped with a topology such that the pairing $\langle\ ,\ \rangle : E^* \times E \to \mathbb{R}$ is continuous. Let K be an admissible compact convex subset of E, and $T : K \to E^*$ continuous. Then there exists a $u_0 \in K$ such that*

$$\langle T(u_0), v - u_0 \rangle \geq 0 \quad \forall\, v \in K. \tag{6.2}$$

Apply Theorem 5.2 as in (i).

(iii) Lions and Stampacchia [36], Stampacchia [66], and Mosco [39]: *let V be an inner product space, X a compact convex subset of V, and $a : V \times V \to \mathbb{R}$ a continuous bilinear form on V. Then for every $v' \in V^*$, there exists a (unique) vector $u \in X$ such that*

$$a(u, u-w) \le \langle v', u-w \rangle \quad \forall\, w \in X. \tag{6.3}$$

Put $E = V$, $S(x) = X$ for $x \in X$,

$$f(u,w) = a(u,-w) - \langle v', -w \rangle \quad \forall\, u, w \in X, \tag{6.4}$$

and apply Theorem 5.2.

(iv) Karamardian [27, Lemma 3.2]: *let X be an admissible compact convex subset of a t.v.s. E, F a topological space, $g : X \to F$ a function, and $\psi : X \times F \to \mathbb{R}$ a function. If for each $y \in F$, $\psi(\cdot, y)$ is quasiconvex on X and the function $(u,v) \mapsto \psi(u, g(v))$ is continuous on $X \times X$, then there exists an $\bar{x} \in X$ such that*

$$\psi(\bar{x}, g(\bar{x})) \le \psi(x, g(\bar{x})) \quad \forall\, x \in X. \tag{6.5}$$

Put $S(x) = X$, $f(x,y) = -\psi(y, g(x))$ for $x, y \in X$, and apply Theorem 5.2.

Note that Karamardian [27] applied (iv) to obtain a variational inequality (v) below, Fan's best approximation theorem, and a solution of the generalized complementarity theorem [27, Theorem 3.1].

(v) Karamardian [27, Corollary 3.1], Juberg and Karamardian [25, Lemma], Park [45, Corollary 1.3]: *let X be an admissible compact convex subset of a t.v.s. E, F a topological space, and $\langle\ ,\ \rangle : F \times E \to \mathbb{R}$ a function which is linear in the second variable. Suppose that $g : X \to F$ is a function such that $(x,y) \mapsto \langle g(x), y \rangle$ is continuous on $X \times E$. Then there exists an $\bar{x} \in X$ such that*

$$\langle g(\bar{x}), y - \bar{x} \rangle \ge 0 \quad \forall\, y \in X. \tag{6.6}$$

Put $S(x) = X$, $f(x,y) = \langle g(x), -y \rangle$ for $x, y \in X$, and apply Theorem 5.2.

(vi) Parida, Sahoo, and Kumar [43, Theorem 3.1], Behera and Panda [1, Theorem 2.2], Siddiqi, Khaliq, and Ansari [63]: *let X be an admissible compact convex subset of a t.v.s. E on which E^* is equipped with a topology such that the pairing $\langle\ ,\ \rangle : E^* \times E \to \mathbb{R}$ is continuous, $T : X \to E^*$ and $\theta : X \times X \to E$ continuous maps such that*

(1) *$\langle T(y), \theta(y,y) \rangle \ge 0$ for all $y \in X$; and*

(2) *for each $y \in X$, the function $\langle Ty, \theta(\cdot, y) \rangle : X \to \mathbb{R}$ is quasiconvex.*

Then there exists an $x_0 \in X$ such that

$$\langle T(x_0), \theta(y, x_0) \rangle \ge 0 \quad \forall\, y \in X. \tag{6.7}$$

Put $S(x) = X$, $f(x,y) = -\langle T(x), \theta(y,x) \rangle$ for $x, y \in X$, and apply Theorem 5.2.

Remarks. (1) Note that the statements (ii)–(vi) are more general than the original ones.

(2) In the frame of the KKM theory, some of (i)–(vi) can be obtained without assuming the admissibility. However, in this section, we want to show the applicability of Theorem 2.1.

(3) In [59], using the Idzik fixed point theorem [23], different versions of results of this section were given.

7. More variational inequalities. From Theorem 3.4 we have the following.

Theorem 7.1. *Let X be a convex subset of a t.v.s. E, K a nonempty compact subset of X, and $S,T : X \multimap K$ a map such that*

(1) *for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$; and*

(2) *$\{\operatorname{Int} S^-(y)\}_{y\in K}$ covers X.*

If $\operatorname{co} K$ is admissible in E, then T has a fixed point.

Remark. Theorem 7.1 for $S = T$ is due to Ben-El-Mechaiekh et al. [3, Theorem 3.2] whenever E is locally convex. Ben-El-Mechaiekh [2] raised a question whether the local convexity can be eliminated. Later, Kim and Tan [30], Zhang [74], Chang and Zhang [8] used Theorem 7.1 for $S = T$ for a locally convex t.v.s. to obtain a type of variational inequalities.

From Theorem 7.1, we deduce the following equilibrium theorem.

Theorem 7.2. *Let X be a convex subset of a t.v.s., $p,q : X\times X \to \mathbb{R}$ functions, and K a nonempty compact subset of X. Suppose that*

(1) *$q(x,x) \le 0$ for all $x \in K$;*

(2) *for each $y \in K$, $\{x \in X : p(x,y) > 0\}$ is open in X;*

(3) *for each $x \in X$, $\{y \in K : q(x,y) > 0\} \supset \operatorname{co}\{y \in K : p(x,y) > 0\}$; and*

(4) *$p(x,y) \le 0$ for all $x \in X$ and $y \in X\backslash K$.*

If $\operatorname{co} K$ is admissible, then there exists an $x_0 \in X$ such that

$$p(x_0,y) \le 0 \quad \forall\, y \in X. \tag{7.1}$$

Proof. Suppose that for each $x \in X$, there exists a $y \in K$ such that $p(x,y) > 0$ and hence $q(x,y) > 0$ by (3). Define $S,T : X \multimap K$ by

$$S(x) = \{y \in K : p(x,y) > 0\}, \qquad T(x) = \{y \in K : q(x,y) > 0\} \tag{7.2}$$

for each $x \in X$. Then

(i) for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$ by (3); and

(ii) for each $x \in X$, there exists a $y \in K$ such that $y \in S(x)$ or $x \in S^-(y) = \{x \in X : p(x,y) > 0\}$. Since $S^-(y)$ is open in X by (2), X is covered by $\{\operatorname{Int} S^-(y)\}_{y\in K}$.

Therefore, by Theorem 7.1, T has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T(\hat{x})$. Hence, $\hat{x} \in K$ and $q(\hat{x},\hat{x}) > 0$. This contradicts (1). Therefore, there exists an $x_0 \in X$ such that

$$p(x_0,y) \le 0 \quad \forall\, y \in K. \tag{7.3}$$

However, this inequality holds for all $y \in X$ because of (4). □

Let $\mathbb{K}$ be the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. In order to obtain variational inequalities related to multimaps, we need the following simple consequence of Berge's theorem.

Lemma 7.3. *Let E be a t.v.s. over $\mathbb{K}$, X a nonempty subset of E, F a topological space, $T : X \multimap F$ an u.s.c. map with compact values, and $\langle\ ,\ \rangle : F\times E \to \mathbb{K}$ a function such that for each $y \in E$, $(f,x) \mapsto \operatorname{Re}\langle f, x-y\rangle$ is l.s.c. on $F\times X$. Then for each $y \in E$, the function*

$$x \longmapsto \inf_{f \in T(x)} \operatorname{Re}\langle f, x - y\rangle \tag{7.4}$$

is l.s.c. on X.

Remark. Lemma 7.3 was recently given by Park and Chen [57, Lemma 2]. From Lemma 7.3, we deduce the following.

Lemma 7.4. *Let E be a t.v.s. over $\mathbb{K}$, F a vector space over $\mathbb{K}$, and $\langle\ ,\ \rangle : F \times E \to \mathbb{K}$ a bilinear function. Let X be a nonempty bounded subset of E such that, for each $z \in F$, $y \mapsto \langle z, y\rangle$ is continuous on X. Suppose that F has the $\eta(F,E)$-topology; that is, the topology of uniform convergence on bounded subsets of E, and $T : X \multimap F$ is u.s.c. with compact values. Then for each $y \in E$, the function*

$$x \longmapsto \inf_{f \in T(x)} \operatorname{Re}\langle f, x - y\rangle \tag{7.5}$$

is l.s.c. on X.

Proof. As in Kum [33, Lemma B], the pairing $\langle\ ,\ \rangle : F \times X \to \mathbb{K}$ is continuous. Therefore, by Lemma 7.3, the proof is completed. □

Remarks. (1) Note that if $F = E^*$, the topological dual of E, then $y \mapsto \langle z, y\rangle$ is obviously continuous for each $z \in E^*$.

(2) Particular forms of Lemma 7.4 were due to Browder [6, Lemma 1], Shih and Tan [62, Lemma 1], Kim and Tan [30, Lemma 2], and Chang and Zhang [8, Lemma 2] with proofs more lengthy than ours.

The following is the main result of Section 7.

Theorem 7.5. *Let X be a bounded convex subset of a t.v.s. E over $\mathbb{K}$, $\mathbb{K}$ a nonempty compact subset of X, F a vector space over $\mathbb{K}$ with the $\eta(F,E)$-topology, and $\langle\ ,\ \rangle : F \times E \to \mathbb{K}$ a bilinear function such that, for each $z \in F$, $y \mapsto \langle z, y\rangle$ is continuous on X. Let $T : X \multimap F$ be an u.s.c. map with compact values, and $\alpha : X \times X \to \mathbb{R}$ a function such that*

(1) *for each $x \in X$, $\alpha(x,x) = 0$, $\alpha(x,\cdot)$ is concave, and $\alpha(\cdot,x)$ is l.s.c.; and*
(2) *for each $x \in X$ and $y \in X \backslash K$,*

$$\inf_{f \in T(x)} \operatorname{Re}\langle f, x - y\rangle + \alpha(x,y) \le 0. \tag{7.6}$$

If $\operatorname{co} K$ is admissible, then there exists an $x_0 \in X$ such that

$$\inf_{f \in T(x_0)} \operatorname{Re}\langle f, x_0 - y\rangle + \alpha(x_0,y) \le 0 \quad \forall\, y \in X. \tag{7.7}$$

Moreover, the set of all solutions x_0 is a closed subset of X. Further if $T(x_0)$ is convex and $\alpha(x_0,\cdot)$ is linear, then there exists an $f_0 \in T(x_0)$ such that

$$\operatorname{Re}\langle f_0, x_0 - y\rangle + \alpha(x_0,y) \le 0 \quad \forall\, y \in X. \tag{7.8}$$

Proof. We use Theorem 7.2 with $p = q$. Let

$$p(x,y) = \inf_{f \in T(x)} \operatorname{Re}\langle f, x - y\rangle + \alpha(x,y). \tag{7.9}$$

Then we have the following:

(i) $p(x,x)=0$ by the property of $\langle\ ,\ \rangle$ and (1).

(ii) For each $y\in\mathbb{K}$, $\{x\in X: p(x,y)>0\}$ is open in X since $x\mapsto p(x,y)$ is l.s.c. on X by Lemma 7.4 and (1).

(iii) For each $x\in X$, $\{y\in K: p(x,y)>0\}$ is convex in $\mathbb{K}$. In fact, for any $y_1,y_2\in\mathbb{K}$ satisfying $p(x,y_1)>0$ and $p(x,y_2)>0$, let $y=ty_1+(1-t)y_2$ for some $t\in(0,1)$. Then

$$\begin{aligned} p(x,y) &= \inf_{f\in T(x)} \operatorname{Re}\langle f, x-(ty_1+(1-t)y_2)\rangle + \alpha(x, ty_1+(1-t)y_2) \\ &\geq t \inf_{f\in T(x)} \operatorname{Re}\langle f, x-y_1\rangle + (1-t)\inf_{f\in T(x)} \operatorname{Re}\langle f, x-y_2\rangle + t\alpha(x,y_1)+(1-t)\alpha(x,y_2) \\ &= tp(x,y_1)+(1-t)p(x,y_2)>0. \end{aligned} \tag{7.10}$$

Note that $y\in K$ by (2).

(iv) For each $y\in X\backslash K$ and $x\in X$, we have $p(x,y)\leq 0$ by (2).

Therefore by Theorem 7.2, there exists an $x_0\in X$ such that

$$p(x_0,y)\leq 0 \quad \forall\, y\in X. \tag{7.11}$$

Moreover, the set of all solutions x_0 is

$$\bigcap_{y\in X}\{x\in X: p(x,y)\leq 0\}, \tag{7.12}$$

which is the intersection of nonempty closed sets by (ii).

To prove the final assertion, suppose that $T(x_0)$ is convex and $\alpha(x_0,\cdot)$ is linear on X. We define a function $g: T(x_0)\times X\to\mathbb{R}$ by

$$g(f,y)=\operatorname{Re}\langle f, x_0-y\rangle+\alpha(x_0,y) \quad \text{for } (f,y)\in T(x_0)\times X. \tag{7.13}$$

Then g is linear in $f\in T(x_0)$ and in $y\in X$. Note that for a given $y\in X$, $f\mapsto g(f,y)$ is continuous on F with the $\eta(F,E)$-topology. Therefore, by the Kneser minimax theorem [32], we have

$$\inf_{f\in T(x_0)}\sup_{y\in X} g(f,y)=\sup_{y\in X}\min_{f\in T(x_0)} g(f,y). \tag{7.14}$$

Since the right-hand side of (7.14) is less than or equal to zero by the first conclusion, we have

$$\inf_{f\in T(x_0)}\sup_{y\in X} g(f,y)\leq 0. \tag{7.15}$$

Since $f\mapsto\sup_{y\in X} g(f,y)$ is l.s.c. and Tx_0 is compact, there exists an $f_0\in T(x_0)$ such that $\sup_{y\in X} g(f_0,y)\leq 0$. This completes the proof. □

Remarks. (1) For a locally convex t.v.s. E, Theorem 7.5 reduces to Chang and Zhang [8, Theorem 1] and Zhang [74, Theorem 3].

(2) In case E is locally convex, $F=E^*$, and $\alpha=0$, Theorem 7.5 reduces to Kim and Tan [30, Theorem 1], which extends Browder [7, Theorem 6].

(3) Note that in some cases we can choose a topology on F different from $\eta(F,E)$ and the assumption on the boundedness of X can be removed from Theorem 7.5. For example, if we choose the topology $\sigma(F,E)$ (see [57]) or in the case of a normed vector space E and $F = E^*$ (see [30, Theorem 3]), we need not to assume the boundedness of X.

For a subset X of a t.v.s. E, the *inward set* $I_X(x)$ of X at $x \in E$ is defined by

$$I_X(x) = \{x + r(y - x) : r > 0,\ y \in X\}, \tag{7.16}$$

and $\bar{I}_X(x)$ denotes its closure.

For a locally convex t.v.s. E, $F = E^*$, and $\alpha = 0$, we have the following from Theorem 7.5 [58].

Theorem 7.6. *Let X be a bounded convex subset of a locally convex t.v.s. E, K a nonempty compact subset of X, and $T : X \to E^*$ a continuous map, where E^* has the $\eta(E^*,E)$-topology. Suppose that for each $x \in X$ and $y \in X\backslash K$, we have*

$$\operatorname{Re}\langle T(x), x - y\rangle \le 0. \tag{7.17}$$

Then there exists an $x_0 \in X$ such that

$$\operatorname{Re}\langle T(x_0), x_0 - y\rangle \le 0 \quad \forall\, y \in \bar{I}_X(x_0). \tag{7.18}$$

Moreover, the set of all solutions x_0 is a closed subset of X.

Remarks. (1) Theorem 7.6 is due to Park and Kang [58, Corollary] and strengthens [30, Corollary 2].

(2) In [58], Theorem 7.6 was used to obtain a far-reaching generalization of fixed point theorems of Kim and Tan [30].

8. The Nash type equilibrium theorems. From the intersection Theorem 4.3, we can deduce the following equivalent form of a generalized Fan type minimax theorem.

Theorem 8.1. *Let $\{X_i\}_{i\in I}$ be a family of compact convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ functions satisfying*

(1) *$g_i(x) \le f_i(x)$ for each $x \in X$;*

(2) *for each $x^i \in X^i$, $x_i \mapsto f_i(x^i, x_i)$ is quasiconcave on X_i; and*

(3) *for each $x_i \in X_i$, $x^i \mapsto g_i(x^i, x_i)$ is l.s.c. on X^i.*

Let $\{t_i\}_{i\in I}$ be a family of real numbers. Then either

(a) *there exists an $i \in I$ and an $x^i \in X^i$ such that*

$$g_i(x^i, x_i) \le t_i \quad \forall\, x_i \in X_i; \tag{8.1}$$

or (b) *there exists an $x \in X$ such that*

$$f_i(x) > t_i \quad \forall\, i \in I. \tag{8.2}$$

Proof. Suppose that (a) does not hold; that is, for any $i \in I$ and any $x^i \in X^i$, there exists an $x_i \in X_i$ such that $g_i(x^i, x_i) > t_i$. Let

$$A_i = \{x \in X : f_i(x) > t_i\}, \qquad B_i = \{x \in X : g_i(x) > t_i\} \tag{8.3}$$

for each $i \in I$. Then

(4) for each $x^i \in X^i$, $\emptyset = B_i(x^i) \subset A_i(x^i)$;

(5) for each $x^i \in X^i$, $A_i(x^i)$ is convex; and

(6) for each $y \in X_i$, $B_i(y)$ is open in X^i.

Therefore, by Theorem 4.3, there exists an $x \in \bigcap_{i\in I} A_i$. This is equivalent to (b). □

Examples. (1) Fan [12, Theorem 2], [13, Theorem 3]: I is finite and $f_i = g_i$ for all $i \in I$. From this, Fan [12, 13] deduced Sion's minimax theorem [65], the Tychonoff fixed point theorem, solutions to systems of convex inequalities, extremum problems for matrices, and a theorem of Hardy-Littlewood-Pólya.

(2) Ma [37, Theorem 3]: $f_i = g_i$ for all $i \in I$.

Remarks. (1) We obtained Theorem 8.1 from Theorem 4.3. As was pointed out by Fan [12] for his case, we can deduce Theorem 4.3 from Theorem 8.1 by considering the characteristic functions of the sets A_i and B_i.

(2) The conclusion of Theorem 8.1 can be stated as follows: if

$$\min_{x_i\in X_i} \sup_{x^i\in X^i} g_i(x^i,x_i) > t_i \quad \forall\, i \in I, \tag{8.4}$$

then (b) holds (see Fan [12, 13]).

From Theorem 4.3, we also obtain the following generalization of the Nash-Ma type equilibrium theorems.

Theorem 8.2. *Let $\{X_i\}$ be a family of compact convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be functions such that*

(1) *$g_i(x) \le f_i(x)$ for each $x \in X$;*

(2) *for each $x^i \in X^i$, $x_i \mapsto f_i(x^i,x_i)$ is quasiconcave on X_i;*

(3) *for each $x^i \in X^i$, $x_i \mapsto g_i(x^i,x_i)$ is u.s.c. on X_i; and*

(4) *for each $x_i \in X_i$, $x^i \mapsto g_i(x^i,x_i)$ is l.s.c. on X^i.*

Then there exists a point $\hat{x} \in X$ such that

$$f_i(\hat{x}) \ge \max_{y_i\in X_i} g_i(\hat{x}^i,y_i) \quad \forall\, i \in I. \tag{8.5}$$

Proof. For any $\varepsilon > 0$, we define

$$\begin{aligned} A_{\varepsilon,i} &= \{x \in X : f_i(x) > \max_{y_i\in X_i} g_i(x^i,y_i) - \varepsilon\}, \\ B_{\varepsilon,i} &= \{x \in X : g_i(x) > \max_{y_i\in X_i} g_i(x^i,y_i) - \varepsilon\} \end{aligned} \tag{8.6}$$

for each i. Then

(1) for each $x^i \in X^i$, $B_{\varepsilon,i}(x^i) \subset A_{\varepsilon,i}(x^i)$;

(2) for each $x^i \in X^i$, $A_{\varepsilon,i}(x^i)$ is convex;

(3) for each $x^i \in X^i$, $B_{\varepsilon,i}(x_i) = \emptyset$ since $x_i \mapsto g_i(x^i,x_i)$ is u.s.c. on the compact space X_i; and

(4) for each $x_i \in X_i$, $B_{\varepsilon,i}(x_i)$ is open since $x^i \mapsto g_i(x^i,x_i)$ is l.s.c. on X^i.

Therefore, by applying Theorem 4.3, we have

$$\bigcap_{i\in I} A_{\varepsilon,i} \neq \emptyset \quad \forall\, \varepsilon > 0. \tag{8.7}$$

Since X is compact, there exists an $\hat{x} \in X$ such that $f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i(\hat{x}^i, y_i)$ for all $i \in I$. □

Examples. (1) In case of $f_i = g_i$ and for a finite I, Theorem 8.2 reduces to Tan et al. [68, Theorem 2.1].

(2) From Theorem 8.2, we obtain the following generalization of the Nash equilibrium theorem due to Ma [37, Theorem 4].

Theorem 8.3. *Let $\{X_i\}_{i\in I}$ be a family of compact convex spaces and, for each $i \in I$, let $f_i : X \to \mathbb{R}$ be a continuous function such that for each $x^i \in X^i$, $x_i \mapsto f_i(x^i, x_i)$ is quasiconcave on X_i. Then there exists a point $\hat{x} \in X$ such that*

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i(\hat{x}^i, y_i) \quad \forall\, i \in I. \tag{8.8}$$

Examples. (1) Nash [41, Theorem 1]: I is finite and X_i are subsets of Euclidean spaces.

(2) Nikaido and Isoda [42, Theorem 3.2]: I is finite.

(3) Fan [13, Theorem 4]: I is finite.

Remark. In view of Theorem 8.2, the continuity of f_i in Theorem 8.3 can be weakened to component-wise u.s.c. and l.s.c., as Sion [65] generalized von Neumann's minimax theorem [70].

Particular forms of results in this section can also be seen in Browder [7].

9. The Tarafdar type equilibrium theorems. In this section, we apply Theorem 3.4 to the existence of equilibrium points and maximal elements of an abstract economy.

An *abstract economy* $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ consists of an index set I of agents, a choice set X_i in a t.v.s. E_i, constraint correspondences $A_i, B_i : X = \prod_{i\in I} X_i \multimap X_i$, and a preference correspondence $P_i : X \multimap X_i$ for each $i \in I$. An *equilibrium point* $x = \{x_i\}_{i\in I} \in X$ is the one satisfying $x_i \in B_i(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for each $i \in I$. We say that $x \in X$ is a *maximal point* of the game $(X_i, P_i)_{i\in I}$ if $P_i(x) = \emptyset$ for each $i \in I$.

Theorem 9.1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy such that, for each $i \in I$,*

(1) *X_i is a convex subset of a t.v.s. E_i and K_i is a nonempty compact subset of X_i;*
(2) *for each $x \in X$, $\operatorname{co} A_i(x) \subset B_i(x) \subset K_i$;*
(3) *$D = \operatorname{co} K \subset \bigcup_{y\in K_i} \operatorname{Int}(A_i^-(y) \cap (P_i^-(y) \cup F_i))$, where $F_i = \{x \in X : A_i(x) \cap P_i(x) = \emptyset\}$; and*
(4) *for each $x = \{x_i\}_{i\in I} \in X$, $x_i \notin \operatorname{co} P_i(x)$.*

If D is admissible, Γ has an equilibrium point in K.

Proof. Let

$$G_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\} \quad \forall\, i \in I. \tag{9.1}$$

For each $i \in I$, we define two maps $S_i, T_i : X \multimap K_i$ by

$$\begin{aligned} S_i(x) &= \begin{cases} A_i(x) \cap \operatorname{co} P_i(x) & \text{if } x \in G_i, \\ A_i(x) & \text{if } x \in F_i, \end{cases} \\ T_i(x) &= \begin{cases} B_i(x) \cap \operatorname{co} P_i(x) & \text{if } x \in G_i, \\ B_i(x) & \text{if } x \in F_i. \end{cases} \end{aligned} \tag{9.2}$$

Then for each $i \in I$ and $x \in X$, we have $\emptyset = \operatorname{co} S_i(x) \subset T_i(x)$; and for each $y \in K_i$, we have

$$\begin{aligned} S_i^-(y) &= \left[\left(A_i^-(y) \cap (\operatorname{co} P_i)^-(y)\right) \cap G_i\right] \cup \left[A_i^-(y) \cap F_i\right] \\ &\supset \left[\left(A_i^-(y) \cap P_i^-(y)\right) \cap G_i\right] \cup \left[A_i^-(y) \cap F_i\right] \\ &= \left[A_i^-(y) \cap P_i^-(y)\right] \cup \left[A_i^-(y) \cap F_i\right] \\ &= A_i^-(y) \cap \left(P_i^-(y) \cup F_i\right), \end{aligned} \tag{9.3}$$

which implies $D = \operatorname{co} K \subset \bigcup_{y \in K_i} \operatorname{Int} S_i^-(y)$ by (3). Therefore, all of the requirements of Theorem 3.4 are satisfied. Hence, there exists an $\bar{x} \in K$ such that $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$. By (4), $\bar{x}_i \notin \operatorname{co} P_i(\bar{x})$. Therefore, $\bar{x}_i \in B_i(\bar{x})$ for each $i \in I$ by the definition of T_i and hence $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$. This shows that $\bar{x}$ is an equilibrium point of Γ. □

Remark. We followed the proof of Tarafdar [69, Theorem 3.1]. If all of E_i's are locally convex, then Theorem 9.1 reduces to Ding et al. [10, Theorem 5] and Husain and Tarafdar [20, Theorem 3.1].

Theorem 9.2. *Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game such that, for each $i \in I$,*

(1) *X_i is a convex subset of a t.v.s. E_i, and K_i is a nonempty compact convex subset of X_i;*
(2) *$K \subset \bigcup_{y \in K_i} \operatorname{Int}_X (P_i^-(y) \cup F_i)$, where $F_i = \{x \in X : P_i(x) = \emptyset\}$; and*
(3) *for each $x = \{x_i\}_{i \in I} \in X$, $x_i \notin \operatorname{co} P_i(x)$.*

If $\operatorname{co} K$ is admissible, then Γ has a maximal element in K.

Proof. For each $i \in I$, define a map $A_i : X \multimap K_i$ by $A_i(x) = K_i$ for $x \in X$. Now we can apply Theorem 9.1 with $A_i = B_i$, and the conclusion follows. □

Remark. If each E_i is locally convex, then Theorem 9.2 reduces to Husain and Tarafdar [20, Theorem 3.2].

10. The Yannelis-Prabhakar type equilibrium theorems. In this section, we point out that some modifications or generalizations of the Yannelis-Prabhakar type equilibrium theorems [73] can also be obtained in the frame of our method in this paper for not-necessarily locally convex t.v.s.

We list some of them as follows.

Theorem 10.1. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy such that, for each $i \in I$,*

(1) *X_i is a convex subset of a t.v.s. E_i and K_i is a nonempty compact subset of X_i;*

(2) *for each* $x \in X$, *there exists a nonempty subset* $B_i(x)$ *such that* $\operatorname{co} B_i(x) \subset A_i(x) \subset K_i$;
(3) $\bar{A}_i : X \multimap K_i$ *defined by* $\bar{A}_i(x) = \overline{A_i(x)}$ *for* $x \in X$ *is a closed map;*
(4) $\bar{P}_i : X \multimap X_i$ *is a closed correspondence, where* $P_i(x)$ *is (possibly empty) convex for each* $x \in X$;
(5) *the set* $W_i = \{x \in X : B_i(x) \cap P_i(x) \neq \emptyset\}$ *is a (possibly empty) closed proper subset of* X; *and*
(6) *for each* $x \in W_i$, $x_i \notin \bar{P}_i(x)$.

If X *is admissible in* E, *then* Γ *has an equilibrium choice* $\hat{x} \in K$; *more precisely, for each* $i \in I$,

$$\hat{x}_i \in \bar{A}_i(\hat{x}), \qquad B_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset. \tag{10.1}$$

Proof. Just follow the proof of Kim et al. [28] or Kim [29] using Theorem 2.1 instead of Himmelberg's. □

Example. If each E_i is locally convex, then Theorem 10.1 reduces to Kim et al. [28] and Kim [29].

Theorem 10.2. *Let* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ *be an abstract economy such that, for each* $i \in I$, *conditions (1)–(3) of Theorem 10.1 and the following hold.*
(4) *for each* $y \in K_i$, $B_i^-(y)$ *is open in* X;
(5) *for each* $y \in X_i$, $P_i^-(y)$ *is open in* X;
(6) *for each* $x \in X$, $x_i \notin \operatorname{co} P_i(x)$; *and*
(7) *the set* $\{x \in X : \operatorname{co} B_i(x) \cap \operatorname{co} P_i(x) = \emptyset\}$ *is paracompact.*

If X *is admissible in* E, *then* Γ *has an equilibrium choice* $\hat{x} \in K$ *as in the conclusion of Theorem 10.1.*

Proof. Just follow the proof of Ding et al. [10, Theorem 4] using Theorem 2.1 instead of Himmelberg's. □

Example. If each E_i is locally convex, then Theorem 10.2 reduces to Ding et al. [10, Theorem 4].

From Theorem 10.2, we immediately have the following.

Theorem 10.3. *Let* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ *be an abstract economy, where* I *is countable, such that, for each* $i \in I$, *the following hold.*
(1) X_i *is a nonempty compact convex subset of a t.v.s.* E_i;
(2) $A_i(x)$ *is convex for all* $x \in X$;
(3) *the map* $\bar{A}_i : X \multimap X_i$ *defined by* $\bar{A}_i(x) = \overline{A_i(x)}$ *for all* $x \in X$ *is closed;*
(4) *for each* $y \in X_i$, $A_i^-(y)$ *is open in* X;
(5) *for each* $y \in X_i$, $P_i^-(y)$ *is open in* X *(*P *may have empty values); and*
(6) $x_i \notin \operatorname{co} P_i(x)$ *for all* $x \in X$.

If X_i *is metrizable for each* $i \in I$ *and if* X *is admissible in* E, *then* Γ *has an equilibrium choice* $\hat{x} \in X$; *that is,* $\hat{x}_i \in \bar{A}_i(x)$ *and* $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$.

Example. If each E_i is locally convex, then Theorem 10.3 reduces to Yannelis and Prabhakar [73, Theorem 6.1].

Remarks. (1) If X_i is not metrizable and X is not admissible, then we may not have an equilibrium in Theorem 10.3. But, even in this case, we have an equilibrium choice for certain "subeconomy."

Since X is compact, from (2) and (3), there exists a continuous selection $f_i : X \to K_i$ of $A_i : X \multimap X_i$, by Lemma 3.1, where K_i is a polytope in X_i. Note that K_i is metrizable. Now define A_i', $P_i' : K \multimap K_i$ by

$$A_i'(x) = A_i(x) \cap K_i, \quad P_i'(x) = P_i(x) \cap K_i, \quad \text{for } x \in K. \tag{10.2}$$

Note that $A_i'(x)$ is nonempty since $f_i(x) \in A_i(x) \cap K_i = A_i'(x)$. It is clear that the "subeconomy" $\Gamma' = (K_i, A_i', P_i')_{i \in I}$ satisfies all of the requirements (1)–(6) replacing (X, X_i, A_i, P_i) by (K, K_i, A_i', P_i'). Moreover, X is admissible as a subset of a locally convex t.v.s. as in the proof of Theorem 3.5. Therefore, there exists an $\hat{x} \in K \subset X$ such that

$$\hat{x}_i \in \bar{A}_i'(x) \subset \bar{A}_i(x), \quad P_i'(\hat{x}) \cap A_i'(\hat{x}) = \varnothing, \quad \forall\, i \in I. \tag{10.3}$$

(2) In [49], different versions of Theorems 8.1, 8.2, 8.3, 9.1, 9.2, 10.1, 10.2, and 10.3 were given by using the Idzik fixed point theorem [56].

(3) Finally, for further applications of Theorem 2.1, the reader may consult [52, 54, 44].

Acknowledgement. The author was supported, in part, by the Non-directed Research Fund, Korea Research Foundation, 1997.

References

[1] A. Behera and G. K. Panda, *A generalization of Browder's theorem*, Bull. Inst. Math. Acad. Sinica **21** (1993), no. 2, 183–186. MR 94i:47106. Zbl 780.90098.

[2] H. Ben-El-Mechaiekh, *The coincidence problem for compositions of set-valued maps*, Bull. Austral. Math. Soc. **41** (1990), no. 3, 421–434. MR 91h:47060. Zbl 688.54030.

[3] H. Ben-El-Mechaiekh, P. Deguire, and A. Granas, *Points fixes et coïncidences pour les fonctions multivoques (applications de type φ et φ^*). II*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 5, 381–384. MR 84m:54042b. Zbl 525.47043.

[4] ———, *Une alternative non linéaire en analyse convexe et applications*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 3, 257–259. MR 84f:52008. Zbl 521.47027.

[5] C. Berge, *Espaces Topologiques: Fonctions Multivoques*, Dunod, Paris, 1959. MR 21#4401. Zbl 088.14703.

[6] F. E. Browder, *A new generalization of the Schauder fixed point theorem*, Math. Ann. **174** (1967), 285–290. MR 36#6991. Zbl 176.45203.

[7] ———, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301. MR 37#4679. Zbl 176.45204.

[8] S. S. Chang and C. J. Zhang, *On a class of generalized variational inequalities and quasi-variational inequalities*, J. Math. Anal. Appl. **179** (1993), no. 1, 250–259. MR 94m:49017. Zbl 803.49011.

[9] S. Y. Chang, *A generalization of KKM principle and its applications*, Soochow J. Math. **15** (1989), no. 1, 7–17. MR 91a:54025. Zbl 747.47033.

[10] X. P. Ding, W. K. Kim, and K. K. Tan, *A selection theorem and its applications*, Bull. Austral. Math. Soc. **46** (1992), no. 2, 205–212. MR 93g:47072. Zbl 762.47030.

[11] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U. S. A. **38** (1952), 121–126. MR 13,858d. Zbl 047.35103.

[12] _____, *Sur un théorème minimax*, C. R. Acad. Sci. Paris **259** (1964), 3925–3928. MR 30#5145. Zbl 138.37304.

[13] _____, *Applications of a theorem concerning sets with convex sections*, Math. Ann. **163** (1966), 189–203. MR 32#8101. Zbl 138.37401.

[14] _____, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), no. 4, 519–537. MR 85i:47060. Zbl 532.47043.

[15] O. Hadžić, *Fixed Point Theory in Topological Vector Spaces*, Univerzitet u Novom Sadu Institut za Matematiku, Novi Sad, 1984. MR 87m:47127. Zbl 576.47030.

[16] P. Hartman and G. Stampacchia, *On some non-linear elliptic differential-functional equations*, Acta Math. **115** (1966), 271–310. MR 34#6355. Zbl 142.38102.

[17] C. J. Himmelberg, *Fixed points of compact multifunctions*, J. Math. Anal. Appl. **38** (1972), 205–207. MR 46#2505. Zbl 225.54049.

[18] C. D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), no. 2, 341–357. MR 92f:52002. Zbl 733.54011.

[19] M. Hukuhara, *Sur l'existence des points invariants d'une transformation dans l'espace fonctionnel*, Jap. J. Math. **20** (1950), 1–4. MR 13,561g. Zbl 041.23801.

[20] T. Husain and E. Tarafdar, *A selection and a fixed point theorem and an equilibrium point of an abstract economy*, Internat. J. Math. Math. Sci. **18** (1995), no. 1, 179–184. MR 95m:90026. Zbl 819.90015.

[21] A. Idzik, *Fixed point theorems for families of functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **26** (1978), no. 11, 913–916 (1979). MR 80d:47088b. Zbl 403.47029.

[22] _____, *Remarks on Himmelberg's fixed point theorems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **26** (1978), no. 11, 909–912 (1979). MR 80d:47088a. Zbl 403.47025.

[23] _____, *Almost fixed point theorems*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 779–784. MR 89i:47108. Zbl 691.47046.

[24] S. M. Im and W. K. Kim, *An application of Himmelberg's fixed point theorem to noncompact optimization problems*, Bull. Inst. Math. Acad. Sinica **19** (1991), no. 2, 147–151. MR 92m:47115. Zbl 754.47037.

[25] R. K. Juberg and S. Karamardian, *On variational type inequalities*, Bull. Un. Mat. Ital. (4) **7** (1973), 336–338. MR 48#12175. Zbl 269.49019.

[26] S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. **8** (1941), 457–459. MR 3,60c. Zbl 061.40304.

[27] S. Karamardian, *Generalized complementarity problem*, J. Optimization Theory Appl. **8** (1971), 161–168. MR 47#10073. Zbl 218.90052.

[28] T. S. Kim, W. K. Kim, and J. S. Lim, *On a new equilibrium existence theorem*, J. Chungcheong Math. Soc. **5** (1992), 143–149.

[29] W. K. Kim, *A new equilibrium existence theorem*, Econom. Lett. **39** (1992), no. 4, 387–390. MR 93i:90131. Zbl 769.90085.

[30] W. K. Kim and K. K. Tan, *A variational inequality in noncompact sets and its applications*, Bull. Austral. Math. Soc. **46** (1992), no. 1, 139–148. MR 93e:47087. Zbl 747.47037.

[31] V. Klee, *Leray-Schauder theory without local convexity*, Math. Ann. **141** (1960), 286–296. MR 24#A1004. Zbl 096.08001.

[32] H. Kneser, *Sur un théorème fondamental de la théorie des jeux*, C. R. Acad. Sci. Paris **234** (1952), 2418–2420. MR 14,301a. Zbl 046.12201.

[33] S. Kum, *A generalization of generalized quasi-variational inequalities*, J. Math. Anal. Appl. **182** (1994), no. 1, 158–164. MR 94m:49012. Zbl 804.49012.

[34] M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), no. 1, 151–201. MR 84k:47049. Zbl 527.47037.

[35] _____, *Réduction du cas multivoque au cas univoque dans les problèmes de coïncidence. [Reduction of the multivalued case to the single-valued case in coincidence problems]*, Fixed Point Theory and Applications (Marseille, 1989) (Harlow), Pitman Res.

Notes Math. Ser., 252, Longman Sci. Tech., 1991, pp. 293–302. MR 92h:47086. Zbl 819.47074.

[36] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493–519. MR 35#7178. Zbl 152.34601.

[37] T. W. Ma, *On sets with convex sections*, J. Math. Anal. Appl. **27** (1969), 413–416. MR 40#676. Zbl 176.42703.

[38] E. Marchi and J. E. Martínez-Legaz, *Some results on approximate continuous selections, fixed points and minimax inequalities*, Fixed Point Theory and Applications (Marseille, 1989) (Harlow), Longman Sci. Tech., 1991, pp. 327–343. MR 93b:47120. Zbl 743.49002.

[39] U. Mosco, *Implicit variational problems and quasi variational inequalities*, Nonlinear Operators and the Calculus of Variations (Summer School, Univ. Libre Bruxelles, Brussels, 1975) (Berlin), Springer, 1976, pp. 83–156. Lecture Notes in Math., Vol. 543. MR 58#23827. Zbl 346.49003.

[40] M. Nagumo, *Degree of mapping in convex linear topological spaces*, Amer. J. Math. **73** (1951), 497–511. MR 13,150b. Zbl 043.17801.

[41] J. Nash, *Non-cooperative games*, Ann. of Math. (2) **54** (1951), 286–295. MR 13,261g. Zbl 045.08202.

[42] H. Nikaidô and K. Isoda, *Note on non-cooperative convex games*, Pacific J. Math. **5** (1955), 807–815. MR 17,506g. Zbl 171.40903.

[43] J. Parida, M. Sahoo, and A. Kumar, *A variational-like inequality problem*, Bull. Austral. Math. Soc. **39** (1989), no. 2, 225–231. MR 90c:49013. Zbl 649.49007.

[44] S. Park, *Fixed points and quasi-equilibrium problems*, Math. Comp. Modelling, to appear.

[45] ———, *Remarks on some variational inequalities*, Bull. Korean Math. Soc. **28** (1991), no. 2, 163–174. MR 93b:49011. Zbl 753.47035.

[46] ———, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, Fixed Point Theory and Applications (Halifax, NS, 1991) (River Edge, NJ), World Sci. Publishing, 1992, pp. 248–277. MR 93j:47087.

[47] ———, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Korean Math. Soc. **31** (1994), no. 3, 493–519. MR 95i:47104. Zbl 829.49002.

[48] ———, *Some existence theorems for two variable functions on topological vector spaces*, Kangweon-Kyungki Math. J. **3** (1995), 11–16.

[49] ———, *Applications of the Idzik fixed point theorem*, Proc. Nonlinear Funct. Anal. Appl. **1** (1996), 21–56.

[50] ———, *Fixed points of acyclic maps on topological vector spaces*, World Congress of Nonlinear Analysts '92, Vol. I–IV (Tampa, FL, 1992) (Berlin), de Gruyter, 1996, pp. 2171–2177. CMP 1 389 243. Zbl 863.47042.

[51] ———, *Fixed points of the better admissible multimaps*, Math. Sci. Res. Hot-Line **1** (1997), no. 9, 1–6. Zbl 915.47042.

[52] ———, *Remarks on a social equilibrium existence theorem of G. Debreu*, Appl. Math. Lett. **11** (1998), no. 5, 51–54. MR 99m:90018.

[53] ———, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), no. 4, 803–829. MR 2000a:47121. Zbl 923.47034.

[54] ———, *Acyclic versions of the von Neumann and Nash equilibrium theorems*, J. Comput. Appl. Math. **113** (2000), no. 1-2, 83–91. CMP 1 735 815. Zbl 991.12607.

[55] S. Park, J. S. Bae, and H. K. Kang, *Geometric properties, minimax inequalities, and fixed point theorems on convex spaces*, Proc. Amer. Math. Soc. **121** (1994), no. 2, 429–439. MR 94i:47093. Zbl 806.47054.

[56] S. Park and M. P. Chen, *Generalized quasivariational inequalities*, Far East J. Math. Sci. **3** (1995), no. 2, 185–190. MR 97b:47071. Zbl 990.75909.

[57] ———, *Generalized variational inequalities of the Hartman-Stampacchia-Browder type*, J. Inequal. Appl. **2** (1998), no. 1, 71–87. CMP 1 671 650. Zbl 890.49004.

[58] S. Park and B. G. Kang, *Generalized variational inequalities and fixed point theorems*, Nonlinear Anal. **31** (1998), no. 1-2, 207–216. MR 98k:47130. Zbl 912.49006.

[59] S. Park and J. A. Park, *The Idzik type quasivariational inequalities and noncompact optimization problems*, Colloq. Math. **71** (1996), no. 2, 287–295. MR 97i:49015. Zbl 874.47036.

[60] S. Park, S. P. Singh, and B. Watson, *Some fixed point theorems for composites of acyclic maps*, Proc. Amer. Math. Soc. **121** (1994), no. 4, 1151–1158. MR 94j:47089. Zbl 806.47053.

[61] J. P. Penot, *Fixed point theorems without convexity*, Bull. Soc. Math. France Mém. **60** (1979), 129–152. MR 81c:47061. Zbl 454.47044.

[62] M. H. Shih and K. K Tan, *Minimax inequalities and applications*, Differential Analysis in Infinite-Dimensional Spaces (Albany, N.Y., 1983) (Providence, R.I.), Amer. Math. Soc., 1986, pp. 45–63. MR 88f:47052. Zbl 588.49007.

[63] A. H. Siddiqi, A. Khaliq, and Q. H. Ansari, *On variational-like inequalities*, Ann. Sci. Math. Québec **18** (1994), no. 1, 95–104. MR 95b:49021. Zbl 807.47054.

[64] S. Simons, *Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems*, Nonlinear Functional Analysis and its Applications, Part 2 (Berkeley, Calif., 1983) (Providence, R.I.), Proc. Sympos. Pure Math., vol. 45, Amer. Math. Soc., 1986, pp. 377–392. MR 87h:49019.

[65] M. Sion, *On general minimax theorems*, Pacific J. Math. **8** (1958), 171–176. MR 20#3506. Zbl 081.11502.

[66] G. Stampacchia, *Variational inequalities*, Theory and Applications of Monotone Operators (Proc. NATO Advanced Study Inst., Venice, 1968) (Gubbio), Edizioni "Oderisi", 1969, pp. 101–192. MR 54#13652.

[67] W. Takahashi, *Existence theorems generalizing fixed point theorems for multivalued mappings*, Fixed Point Theory and Applications (Marseille, 1989) (Harlow), Longman Sci. Tech., 1991, pp. 397–406. MR 92m:54078. Zbl 760.47029.

[68] K. K. Tan, J. Yu, and X. Z. Yuan, *Existence theorems of Nash equilibria for non-cooperative n-person games*, Internat. J. Game Theory **24** (1995), no. 3, 217–222. MR 97e:90120. Zbl 837.90126.

[69] E. Tarafdar, *A fixed point theorem and equilibrium point of an abstract economy*, J. Math. Econom. **20** (1991), no. 2, 211–218. MR 91k:90037. Zbl 718.90014.

[70] J. von Neumann, *Zur Theorie der Gesellschaftsspiele*, Math. Ann. **100** (1928), 295–320.

[71] ———, *Üeber ein öekonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergeb. Math. Kolloq. **8** (1937), 73–83. Zbl 017.03901.

[72] H. Weber, *Compact convex sets in non-locally convex linear spaces*, Note Mat. **12** (1992), 271–289. MR 95a:46008. Zbl 846.46004.

[73] N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econom. **12** (1983), no. 3, 233–245. MR 87h:90061a. Zbl 536.90019.

[74] C. J. Zhang, *Generalized variational inequalities and generalized quasi-variational inequalities*, Appl. Math. Mech. **14** (1993), no. 4, 315–325 (Chinese ed.), Translation: Appl. Math. Mech. [Applied Mathematics and Mechanics English Edition] **14** (1993), no. 4, 333-314. MR 94g:49026. Zbl 780.49010.

Sehie Park: Department of Mathematics, Seoul National University, Seoul 151-742, Korea

E-mail address: shpark@math.snu.ac.kr