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ON AN INCLUSION THEOREM

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ABSTRACT. We have established a relation between $\theta-|R,p_n|_k$ and $\theta-|R,q_n|_k$ summability methods, $k>1$, which generalizes a result of Sunouchi (1949) on $|R,p_n|$ and $|R,q_n|$ summability methods.

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1. Introduction. Let (θ_n) be a sequence of positive numbers and let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). We say that the series $\sum a_n$ is summable $\theta-|C,0|_k$, $k\geq 1$, if

$$\sum_{n=1}^{\infty}\theta_n^{k-1}\left|a_n\right|^k<\infty. \tag{1.1}$$

If we take $\theta_n=n$, then $\theta-|C,0|_k$ summability is the same as $|C,0|_k$ summability. Let (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n}p_v\longrightarrow\infty\quad\text{as } n\longrightarrow\infty,\ \left(P_{-i}=p_{-i}=0,\ i\geq 1\right). \tag{1.2}$$

The sequence-to-sequence transformation

$$t_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v s_v \tag{1.3}$$

defines the sequence (t_n) of the (R,p_n) mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [3]). We say that the series $\sum a_n$ is summable $\theta-|R,p_n|_k$, $k\geq 1$, if

$$\sum_{n=1}^{\infty}\theta_n^{k-1}\left|t_n-t_{n-1}\right|^k<\infty. \tag{1.4}$$

In the special case when $\theta_n=n$ (respectively, $k=1$), $\theta-|R,p_n|_k$ summability is the same as $|R,p_n|_k$ (respectively, $|R,p_n|$) summability. The (R,p_n) mean is said to be absolutely kth power conservative if $|C,0|_k\Rightarrow|R,p_n|_k$. We say that the (R,p_n) mean is absolutely kth power θ-conservative if $\theta-|C,0|_k\Rightarrow\theta-|R,p_n|_k$.

A summability method P is said to be stronger than another summability method Q, if the summability of a series by the method Q implies its summability by the method P. If, in addition, the method P sums the series to the same sum as that obtained by Q, the method P is said to include the method Q. The following theorem is known.

Theorem 1.1 (see [4]). *Suppose that $p_n > 0$, $P_n \to \infty$ and suppose similarly that $q_n > 0$, $Q_n \to \infty$. In order that*

$$|R, p_n| \Longrightarrow |R, q_n| \tag{1.5}$$

it is sufficient that

$$\frac{q_n P_n}{p_n Q_n} = O(1). \tag{1.6}$$

In 1950, while reviewing [4], Bosanquet [2], observed that (1.6) is also necessary for the conclusion and completed Theorem 1.1 in necessary and sufficient form.

2. The main result.

The aim of this paper is to generalize Bosanquet's result for $\theta - |R, p_n|_k$ and $\theta - |R, q_n|_k$ summability, where $k \geq 1$. Now, we shall prove the following theorem.

Theorem 2.1. *Let $k > 1$. In order that*

$$\theta - |R, p_n|_k \Longrightarrow \theta - |R, q_n|_k \tag{2.1}$$

should hold (1.6) is necessary. If we suppose that (R, q_n) is "absolutely kth power θ-conservative," i.e.,

$$\theta - |C, 0|_k \Longrightarrow \theta - |R, q_n|_k, \tag{2.2}$$

then (1.6) is also sufficient.

It should be noted that, if we take $k = 1$ in this theorem, then we get Bosanquet's result. Also if we take $\theta_n = n$, then we get another result related to $|R, p_n|_k$ and $|R, q_n|_k$ summability methods.

We need the following lemma for the proof of our theorem.

Lemma 2.2 (see [1]). *Let $k \geq 1$ and let $A = (a_{nv})$ be an infinite matrix. In order that $A \in (l^k, l^k)$ it is necessary that*

$$a_{nv} = O(1) \quad (\text{all } n, v). \tag{2.3}$$

3. Proof of the theorem

Necessity. For the proof of the necessity, we consider the series-to-series version of (1.3), i.e., for $n \geq 1$, let

$$b_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \tag{3.1}$$

$$c_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v. \tag{3.2}$$

If we consider (3.1), we have

$$P_{n-1} a_n = \frac{P_n P_{n-1}}{p_n} b_n - \frac{P_{n-1} P_{n-2}}{p_{n-1}} b_{n-1}. \tag{3.3}$$

Hence

$$a_n = \frac{P_n}{p_n} b_n - \frac{P_{n-2}}{p_{n-1}} b_{n-1}. \tag{3.4}$$

A simple calculation shows that for $n \geq 1$,

$$c_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{b_v}{p_v} (Q_{v-1} P_v - P_{v-1} Q_v) + \frac{q_n P_n}{p_n Q_n} b_n. \tag{3.5}$$

From this we can write down at once the matrix A that transforms $(\theta_n^{1-1/k} b_n)$ into $(\theta_n^{1-1/k} c_n)$. Thus every $\theta - |R, p_n|_k$ summable series $\theta - |R, q_n|_k$ summable if and only if $A \in (l^k, l^k)$. By the lemma, it is necessary that the diagonal terms of A must be bounded, which gives that (1.6) must hold.

Sufficiency. Let $c_{n,1}$ denote the sum on the right-hand side of (3.5) and let $c_{n,2}$ denote the second term on the right-hand side of (3.5). Suppose the conditions are satisfied. Then it is enough to show that if

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |b_n|^k < \infty, \tag{3.6}$$

we have

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |c_{n,i}|^k < \infty \quad (i = 1, 2). \tag{3.7}$$

For $i = 2$ this is an immediate corollary of (1.6). Now consider $i = 1$. We have

$$Q_{v-1} P_v - P_{v-1} Q_v = -P_v q_v + p_v Q_v = O(p_v Q_v) \tag{3.8}$$

by (1.6). Thus

$$c_{n,1} = O\left(\frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} Q_v |b_v| \right). \tag{3.9}$$

Now the assumption (2.2) can be stated in the form that if $\sum b_v \in \theta - |C, 0|_k$ and if

$$d_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} b_v, \tag{3.10}$$

then

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |d_n|^k < \infty. \tag{3.11}$$

Now, define

$$b'_v = \begin{cases} 0 & (v = 1), \\ |b_{v-1}| & (v \geq 2). \end{cases} \tag{3.12}$$

If $\sum b_n \in \theta - |C, 0|_k$, then $\sum b'_n \in \theta - |C, 0|_k$ so applying (2.2) with b_n replaced by b'_n (and making an obvious change of variable in the sum defining d'_n below) we see that if

$$d'_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_v |b_v|, \tag{3.13}$$

then

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |d_n'|^k < \infty. \tag{3.14}$$

Hence (3.7) (with $i = 1$) follows from (3.9). This completes the proof of the theorem. □

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