

UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. We study some integral operators and determine conditions for the univalence of these integral operators.

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1. Introduction. Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in \mathbb{C}; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the class of the functions $f \in A$ which are univalent in U .

Many authors studied the problem of integral operators which preserve the class S . In this sense an important result is due to Pfaltzgraff [4].

THEOREM 1.1 [4]. *If f is univalent in U , α a complex number and $|\alpha| \leq 1/4$, then the function*

$$G_\alpha(z) = \int_0^z [f'(\xi)]^\alpha d\xi \quad (1.1)$$

is univalent in U .

THEOREM 1.2 [3]. *If the function $g \in S$ and α is a complex number, $|\alpha| \leq 1/(4n)$, then the function defined by*

$$G_{\alpha,n}(z) = \int_0^z [g'(u^n)]^\alpha du \quad (1.2)$$

is univalent in U for all positive integer n .

2. Preliminary results. We need the following theorems.

THEOREM 2.1 [2]. *Let α be a complex number, $\operatorname{Re} \alpha > 0$ and $f \in A$. If*

$$\frac{1 - |z|^{2\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (2.1)$$

for all $z \in U$, then for any complex number β , $\operatorname{Re} \beta \geq \operatorname{Re} \alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du \right]^{1/\beta} \quad (2.2)$$

is in the class S .

THEOREM 2.2 [1]. *If the function g is regular in U and $|g(z)| < 1$ in U , then for all $\xi \in U$ and $z \in U$ the following inequalities hold:*

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \bar{z}\xi} \right|, \tag{2.3}$$

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}, \tag{2.4}$$

the equalities hold only in the case $g(z) = \epsilon(z + u)/(1 + \bar{u}z)$, where $|\epsilon| = 1$ and $|u| < 1$.

REMARK 2.3 [1]. For $z = 0$, from inequality (2.3)

$$\left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)} \right| \leq |\xi| \tag{2.5}$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}. \tag{2.6}$$

Considering $g(0) = a$ and $\xi = z$,

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|}. \tag{2.7}$$

for all $z \in U$.

SCHWARZ LEMMA [1]. *If the function g is regular in U , $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold:*

$$|g(z)| \leq |z| \tag{2.8}$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (2.8) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3. Main results

THEOREM 3.1. *Let α, γ be complex numbers, $\text{Re } \alpha = a > 0$ and $g \in A$.*

If

$$\left| \frac{g''(z)}{g'(z)} \right| \leq \frac{1}{n} \tag{3.1}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{n + 2a}{2} \left(\frac{n + 2a}{n} \right)^{n/2a}, \tag{3.2}$$

then for any complex number β , $\text{Re } \beta \geq a$, the function

$$G_{\beta, \gamma, n}(z) = \left\{ \beta \int_0^z u^{\beta-1} [g'(u^n)]^\gamma du \right\}^{1/\beta} \tag{3.3}$$

is in the class S for all $n \in N^* - \{1\}$.

PROOF. Let us consider the function

$$f(z) = \int_0^z [g'(u^n)]^y du. \tag{3.4}$$

The function

$$p(z) = \frac{1}{|y|} \frac{f''(z)}{f'(z)}, \tag{3.5}$$

where the constant $|y|$ satisfies the inequality (3.2), is regular in U .

From (3.4) and (3.5), we obtain

$$p(z) = \frac{y}{|y|} \left[\frac{nz^{n-1}g''(z^n)}{g'(z^n)} \right]. \tag{3.6}$$

Using (3.1) and (3.6) we obtain

$$|p(z)| < 1 \tag{3.7}$$

for all $z \in U$. For $z = 0$ we have $p(0) = 0$.

From (3.6) and Schwarz lemma it results that

$$\frac{1}{|y|} \left| \frac{f''(z)}{f'(z)} \right| \leq |z|^{n-1} \leq |z| \tag{3.8}$$

for all $z \in U$, and hence

$$\left(\frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq |y| \left(\frac{1 - |z|^{2a}}{a} \right) |z|^n. \tag{3.9}$$

Let us consider $Q : [0, 1] \rightarrow R$, $Q(x) = ((1 - x^{2a})/a)x^n$, $x = |z|$. We have

$$Q(x) \leq \frac{2}{n + 2a} \left(\frac{n}{n + 2a} \right)^{n/2a} \tag{3.10}$$

for all $x \in [0, 1]$. From (3.2), (3.9), and (3.10) we obtain

$$\left(\frac{1 - |z|^{2a}}{a} \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{3.11}$$

for all $z \in U$. Then, from (3.11) and Theorem 2.1 it follows that the function $G_{\beta,y,n}$ is in the class S . □

THEOREM 3.2. Let α, y be complex numbers, $Re \alpha = b > 0$ and the function $g \in A$, $g(z) = z + a_2z^2 + \dots$. If

$$\left| \frac{g''(z)}{g'(z)} \right| < 1 \tag{3.12}$$

for all $z \in U$ and the constant $|y|$ satisfies the condition

$$|y| \leq \frac{1}{\max_{|z| \leq 1} [((1 - |z|^{2b})/b)|z|((|z| + 2|a_2|)/(1 + 2|a_2||z|))]} \tag{3.13}$$

then for any complex number β , $\operatorname{Re} \beta \geq b$ the function

$$G_{\beta, \gamma}(z) = \left\{ \beta \int_0^z u^{\beta-1} [g'(u)]^\gamma du \right\}^{1/\beta} \quad (3.14)$$

is in the class S .

PROOF. Let us consider the function

$$f(z) = \int_0^z [g'(u)]^\gamma du. \quad (3.15)$$

The function

$$h(z) = \frac{1}{|\gamma|} \frac{f''(z)}{f'(z)}, \quad (3.16)$$

where the constant $|\gamma|$ satisfies the inequality (3.13), is regular in U .

From (3.15) and (3.16) we have

$$h(z) = \frac{\gamma}{|\gamma|} \frac{g''(z)}{g'(z)}. \quad (3.17)$$

Using (3.12) and (3.17) we obtain

$$|h(z)| < 1, \quad (3.18)$$

for all $z \in U$ and $|h(0)| = 2|a_2|$.

Remark 2.3 applied to the function h gives

$$\frac{1}{|\gamma|} \left| \frac{f''(z)}{f'(z)} \right| \leq \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \quad (3.19)$$

for all $z \in U$.

From (3.19) we obtain

$$\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \frac{1 - |z|^{2b}}{b} |z| \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \quad (3.20)$$

for all $z \in U$. Hence, we have

$$\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\gamma| \max_{|z| \leq 1} \left[\frac{1 - |z|^{2b}}{b} |z| \frac{|z| + 2|a_2|}{1 + 2|a_2||z|} \right]. \quad (3.21)$$

From (3.13) and (3.21) we obtain

$$\frac{1 - |z|^{2b}}{b} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (3.22)$$

for all $z \in U$. From Theorem 2.1, it follows that the function $G_{\beta, \gamma}$ defined by (3.14) is in the class S . \square

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