Internat. J. Math. & Math. Sci.
Vol. 24, No. 6 (2000) 403–418
S0161171200002180


# A COMMUTATOR THEOREM FOR FRACTIONAL INTEGRALS IN SPACES OF HOMOGENEOUS TYPE

**JORGE J. BETANCOR**

(Received 22 October 1998)

ABSTRACT. We give a new proof of a commutator theorem for fractional integrals in spaces of homogeneous type.

Keywords and phrases. Commutator fractional integrals, homogeneous type spaces.

2000 Mathematics Subject Classification. Primary 42B20.

**1. Introduction.** Bramanti and Cerutti [3] and Bramanti [2] extended a classical commutator theorem for fractional integrals due to Chanillo [5] to the context of spaces of homogeneous type. In [3] Bramanti and Cerutti follow an idea contained in [7], based in holomorphic families of operators, used to study the $L^p$ boundedness of singular integrals in Euclidean spaces. In [2] Bramanti investigated the boundedness of the commutator of certain integral operators having positive kernels. A fractional integral appears as a particular case. Bramanti deduces the boundedness of the commutator from a suitable inequality that involves the maximal sharp function. In this paper, we give a different proof to the commutator theorem for fractional integrals in spaces of homogeneous type. We follow the original proof of Chanillo [5] and a good $\lambda$ inequality is essential.

We firstly recall the main definitions needed in the paper (see [8, 9, 11]). $(X,\delta,\mu)$ will be a space of homogeneous type. That is, $X$ is a nonvoid set, $\delta$ is a quasidistance on $X$, i.e., $\delta : X\times X \to [0,\infty)$ is a function satisfying the following properties:

(i) $\delta(x,y) = 0$ if and only if $x = y$,

(ii) $\delta(x,y) = \delta(y,x)$, for every $x,y \in X$, and

(iii) there exists a positive constant $k$ such that for every $x,y,z \in X$

$$\delta(x,y) \le k\big(\delta(x,z)+\delta(z,y)\big), \tag{1.1}$$

and $\mu$ is a positive regular measure on $X$ defined on a $\sigma$-algebra of subsets of $X$ which contains the open sets (in the topology induced by the uniform structure associated to $\delta$) and the ball $B(x,r) = \{y \in X : \delta(x,y) < r\}$, for every $x \in X$ and $r > 0$, and that satisfies the doubling condition: there exists $A > 0$ for which

$$0 < \mu\big(B(x,2r)\big) \le A\mu\big(B(x,r)\big), \tag{1.2}$$

for each $x \in X$ and $r > 0$. Note that if $X$ has more than one element, then $k \ge 1$. The trivial case $k < 1$ is not considered in this paper.

There are many interesting examples of spaces of homogeneous type. For instance, any $C^\infty$ compact Riemannian manifold with the Riemaniann metric and volume and the boundary of any bounded Lipschitz domain in $\mathbb{R}^n$ with the induced Euclidean metric and the Lebesgue measure are spaces of homogeneous type.

A space of homogeneous type is said to be normal if there exist positive constants $A_1$ and $A_2$ such that for every $x \in X$,

$$\begin{gathered} A_1 r \le \mu(B(x,r)), \quad \text{when } 0 < r < R_x, \\ \mu(B(x,r)) \le A_2 r, \quad \text{if } r \ge r_x, \end{gathered} \tag{1.3}$$

where

$$\begin{gathered} R_x = \begin{cases} \infty, & \text{if } \mu(X) = \infty, \\ \inf\{r > 0 : B(x,r) = X\}, & \text{if } \mu(X) < \infty, \end{cases} \\ r_x = \begin{cases} 0, & \text{if } \mu(\{x\}) = 0, \\ \sup\{r > 0 : B(x,r) = \{x\}\}, & \text{if } \mu(\{x\}) > 0. \end{cases} \end{gathered} \tag{1.4}$$

Sufficient conditions, in order that a space $(X,\delta,\mu)$ of homogeneous type admits a quasidistance $d$ that is equivalent to $\delta$ and such that $(X,d,\mu)$ is normal, are given in [14, Lemma 22].

A space of homogeneous type is of order $\rho$, $0 < \rho \le 1$, if there is a positive constant $C$ such that for every $x,y,z \in X$

$$|\delta(x,z) - \delta(y,z)| \le C\delta(x,y)^\rho \left(\max\{\delta(x,z),\delta(y,z)\}\right)^{1-\rho}. \tag{1.5}$$

For each $1 \le p \le \infty$, $L^p(X,\mu)$ and $\|\cdot\|_p$ have the usual meanings. We say that a complex valued measurable function $f$ on $X$ is in $L^p_{\text{loc}}(X,\mu)$, $1 \le p < \infty$, if $\int_{B(x,r)} |f(x)|^p d\mu(x) < \infty$, for every $r > 0$ and for some (and then for all) $x \in X$.

Let $b \in L^1_{\text{loc}}(X,\mu)$. We define $b_\epsilon(x)$, with $x \in X$ and $\epsilon > 0$, as the mean value

$$b_\epsilon(x) = \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon)} b(y)\,d\mu(y). \tag{1.6}$$

If $1 \le p < \infty$ we will say that a function $b \in L^p_{\text{loc}}(X,\mu)$ is in $BMO_p$ if and only if,

$$\|b\|_{*,p} =: \left\| \sup_{\epsilon>0} \left\{ \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon)} |b(y) - b_\epsilon(x)|^p d\mu(y) \right\}^{1/p} \right\|_\infty < \infty. \tag{1.7}$$

We define on $BMO_p$ a “norm” as follows:

$$\|b\|^{(p)} = \begin{cases} \|b\|_{*,p}, & \text{if } \mu(X) = \infty, \\ \|b\|_{*,p} + \left| \int_X b(x)\,d\mu(x) \right|, & \text{if } \mu(X) < \infty. \end{cases} \tag{1.8}$$

When $\mu(X) < \infty$, $(BMO_p, \|\cdot\|^{(p)})$ is a Banach space. If $\mu(X) = \infty$, then we introduce in $BMO_p$ the following relation: let $b_1$ and $b_2$ be in $BMO_p$,

$$b_1 \sim b_2 \iff \text{there exists } C \in \mathbb{C} \text{ such that } b_1 - b_2 = C. \tag{1.9}$$

It is clear that if $b_1, b_2 \in BMO_p$ and $b_1 \sim b_2$, then $\|b_1\|^{(p)} = \|b_2\|^{(p)}$. The quotient space $BMO_p/\sim$ will be denoted again by $BMO_p$ and by considering on it the norm induced by $\|\cdot\|^{(p)}$, $BMO_p$ is a Banach space. As it was proved by Coifman and Weiss [9, page 594], if $1 \le p, q < \infty$, the spaces $BMO_p$ and $BMO_q$ coincide and the norms $\|\cdot\|^{(p)}$ and $\|\cdot\|^{(q)}$ are equivalent. In the sequel, as usual, we will denote by $BMO$ the space $BMO_p$, $1 \le p < \infty$.

Let $0 \le \alpha < 1$. The fractional maximal function $M_\alpha f$ of $f \in L^1_{\text{loc}}(X,\mu)$ is defined by

$$(M_\alpha f)(x) = \sup_{B: x \in B} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)|\, d\mu(y), \quad x \in X. \tag{1.10}$$

Here, for each $x \in X$, the supremum is taken over all those $B$ balls in $X$ containing to $x$. As usual we denote by $M$ the maximal operator $M_0$.

The fractional integral of order $\alpha$ of $f$, $I_\alpha f$, is given by

$$(I_\alpha f)(x) = \int_{X-\{x\}} \frac{f(y)}{\delta(x,y)^{1-\alpha}} d\mu(y). \tag{1.11}$$

In this paper, we study the boundedness of the commutator $[I_\alpha, b]$ of the fractional integral $I_\alpha$ and the multiplier operator associated to a measurable function $b$ on $X$ defined through

$$[I_\alpha, b](f) = bI_\alpha(f) - I_\alpha(bf). \tag{1.12}$$

Throughout this paper, for every $1 \le p < \infty$, we will denote by $p'$ the conjugate of $p$. By $C$ we will always represent a positive constant not necessarily the same in each occurrence.

The following theorem is the main result of the paper.

**Theorem 1.1.** *Let $0 < \alpha < 1$, $0 \le \rho < 1$, $1 < p < 1/\alpha$, and $1/q = 1/p - \alpha$. Assume that $(X,\delta,\mu)$ is a normal space of homogeneous type of order $\rho$ such that $\mu(\{x\}) = 0$, $x \in X$. Then the commutator operator $[I_\alpha, b]$ is bounded from $L^p(X,\mu)$ into $L^q(X,\mu)$ provided that $b \in BMO$.*

Let now $(X,\delta,\mu)$ be a normal space of homogeneous type and of order $\rho \in (0,1)$, such that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$, for every $x \in X$. Gatto, Segovia, and Vagi [10] defined, for every $0 < \alpha < 1$, a function $\delta_\alpha$ on $X \times X$ as follows:

$$\delta_\alpha(x,y) = \left( \int_0^\infty t^{\alpha-1} s(x,y,t)\, dt \right)^{1/\alpha - 1}, \quad \text{for } x \ne y, \tag{1.13}$$

where $s$ represents a symmetric approximation to the identity in the sense of Coifman, and

$$\delta_\alpha(x,y) = 0, \quad \text{for } x = y. \tag{1.14}$$

In [10, Lemma 2] it is proved that, for every $0 < \alpha < 1$, $\delta_\alpha$ is a quasidistance equivalent to $\delta$. Moreover, for each $0 < \alpha < 1$, $(X, \delta_\alpha, \mu)$ is a normal space of homogeneous type of order $\rho$.

Also these authors introduced the fractional integral $\tilde{I}_\alpha$ of order $\alpha \in (0,1)$ through

$$(\tilde{I}_\alpha f)(x) = \int_{X-\{x\}} \frac{f(y)}{\delta_\alpha(x,y)^{1-\alpha}} d\mu(y). \tag{1.15}$$

If we represent by $BMO(\alpha)$ the $BMO$-space associated to the quasidistance $\delta_\alpha$, $0<\alpha<1$, it is immediately deduced from Theorem 1.1 the following commutator theorem for the fractional integral $\tilde{I}_\alpha$.

**Corollary 1.2.** *Assume that $(X,\delta,\mu)$ is a normal space of homogeneous type and of order $\rho \in (0,1)$, such that $\mu(X)=\infty$ and $\mu(\{x\})=0$, for every $x\in X$. Let $0<\alpha<1$. Then the commutator operator $[\tilde{I}_\alpha, b]$ defined by*

$$[\tilde{I}_\alpha, b](f) = b\tilde{I}_\alpha(f) - \tilde{I}_\alpha(bf), \tag{1.16}$$

*is a bounded operator from $L^p(X,\mu)$ into $L^q(X,\mu)$ provided that $1<p<1/\alpha$, $1/q = 1/p-\alpha$ and $b\in BMO(\alpha)$.*

## 2. The proof of the commutator theorem.

In this section, we will prove Theorem 1.1. To see that result we previously establish six lemmas.

Boundedness of the fractional integral $I_\alpha$ was studied in [11, Theorem 1] and [12, Theorems 2.2 and 2.4].

**Lemma 2.1** (see [11, Theorem 1]). *Let $1<p<1/\alpha$ and $1/q=1/p-\alpha$. If $(X,\delta,\mu)$ is a normal space of homogeneous type, then*

(i) *$I_\alpha$ maps continuously $L^p(X,\mu)$ into $L^q(X,\mu)$.*

(ii) *There exists $C_1>0$ such that*

$$\mu(\{x\in X : |I_\alpha(f)(x)| > \lambda\}) \le C_1\left(\frac{\|f\|_1}{\lambda}\right)^{1/1-\alpha}, \tag{2.1}$$

*for every $f\in L^1(X,\mu)$ and $\lambda>0$.*

Kokilashvili and Kufner [12, Theorem 3.2] proved a weighted version of [11, Theorem 1].

Kokilashvili and Kufner [12] established weighted inequalities for the maximal fractional operator $M_\alpha$. Also Wheeden [15] and Bernardis and Salinas [1] gave characterizations for the pairs of weight functions for which $M_\alpha$ is a bounded operator between the corresponding weighted $L^p$-spaces.

The following result can be easily infered from [15, Theorem 4] (also from [12, Proposition A]).

**Lemma 2.2.** *Let $1<p<1/\alpha$ and $1/q=1/p-\alpha$. Then $M_\alpha$ is a bounded operator from $L^p(X,\mu)$ into $L^q(X,\mu)$.*

We now define the auxiliar operator $C(b,f)$ on $X$ as follows:

$$C(b,f)(x) = \sup_{\epsilon>0}\left|\int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f(y)\,d\mu(y)\right|, \quad x\in X, \tag{2.2}$$

where $b$ and $f$ are measurable complex functions on $X$.

Next a useful weak type inequality for the operator $C(b,f)$ is established.

**Lemma 2.3.** *Assume that* $(X,\delta,\mu)$ *is a normal space of homogeneous type. Let* $1 < p < 1/\alpha$. *If* $f \in L^p(X,\mu)$ *and* $b \in L^{p'}(X,\mu)$, *then*

$$\mu(\{x \in X : C(b,f)(x) > \lambda\}) \le C_0\left(\frac{\|b\|_{p'}\|f\|_p}{\lambda}\right)^{1/1-\alpha}, \quad \text{for every } \lambda > 0. \tag{2.3}$$

**Proof.** It is not hard to see that

$$\begin{aligned} C(b,f)(x) &\le \sup_{\epsilon>0}\int_{X\setminus B(x,\epsilon)} \frac{|b(y)||f(y)|}{\delta(x,y)^{1-\alpha}}d\mu(y) \\ &\quad + \sup_{\epsilon>0}|b_\epsilon(x)|\int_{X\setminus B(x,\epsilon)} \frac{|f(y)|}{\delta(x,y)^{1-\alpha}}d\mu(y) \\ &\le I_\alpha(|bf|)(x) + I_\alpha(|f|)(x)M(b)(x), \quad x \in X. \end{aligned} \tag{2.4}$$

Moreover Holder inequality and Lemmas 2.1 and 2.2 lead to

$$\begin{aligned} &\int_X M(b)(x)^{1/1-\alpha} I_\alpha(|f|)(x)^{1/1-\alpha} d\mu(x) \\ &\quad \le \left(\int_X M(b)(x)^{p'}d\mu(x)\right)^{1/p'(1-\alpha)} \left(\int_X I_\alpha(|f|)(x)^{p'/p'(1-\alpha)-1}d\mu(x)\right)^{1-(1/p'(1-\alpha))} \\ &\quad \le C\|b\|_{p'}^{1/1-\alpha}\|f\|_p^{1/1-\alpha}. \end{aligned} \tag{2.5}$$

Hence if $\lambda > 0$, then

$$\mu(\{x \in X : M(b)(x)I_\alpha(|f|)(x) > \lambda\}) \le C\left(\frac{\|b\|_{p'}\|f\|_p}{\lambda}\right)^{1/1-\alpha}. \tag{2.6}$$

Also by taking into account Lemma 2.1 we have

$$\begin{aligned} \mu\left(\{x \in X : I_\alpha(|bf|)(x) > \lambda\}\right) &\le C\left(\frac{\|bf\|_1}{\lambda}\right)^{1/1-\alpha} \\ &\le C\left(\frac{\|b\|_{p'}\|f\|_p}{\lambda}\right)^{1/1-\alpha}, \quad \lambda > 0. \end{aligned} \tag{2.7}$$

Now to finish the proof of this lemma it is sufficient to combine (2.4), (2.6), and (2.7). □

**Lemma 2.4.** *Assume that* $(X,\delta,\mu)$ *is a normal space of homogeneous type such that* $\mu(\{x\}) = 0$, $x \in X$. *Let* $0 < \alpha < 1$, $1 < p < 1/\alpha$, $0 < \beta < 1/k$, *and* $d$, $\gamma > 0$. *Let* $b \in BMO$ *and* $f$ *be a measurable function. Then*

$$d^\gamma \int_{X\setminus B(x,d)} \frac{|b(y)-b_d(x)|}{\delta(x,y)^{1+\gamma-\alpha}}|f(y)|d\mu(y) \le C\left(M_{\alpha p}(|f|^p)(x_0)\right)^{1/p}\|b\|_{*,p'}, \tag{2.8}$$

*provided that* $\delta(x,x_0) \le \beta d$. *Here* $C$ *is a constant that does not depend on* $d$.

**Proof.** Suppose that $\mu(X)=\infty$. If $\mu(X)<\infty$ we can proceed in a similar way. Holder inequality implies that

$$\begin{aligned} d^\gamma \int_{X\setminus B(x,d)} & \frac{|b(y)-b_d(x)|}{\delta(x,y)^{1+\gamma-\alpha}}|f(y)|d\mu(y) \\ & \le \left(d^\gamma \int_{X\setminus B(x,d)} \frac{|b(y)-b_d(x)|^{p'}}{\delta(x,y)^{1+\gamma}}d\mu(y)\right)^{1/p'} \\ & \quad \times \left(d^\gamma \int_{X\setminus B(x,d)} \frac{|f(y)|^p}{\delta(x,y)^{1+\gamma-p\alpha}}d\mu(y)\right)^{1/p}. \end{aligned} \tag{2.9}$$

Since $\mu$ is doubling we can write for every $x\in X$ and $j\in\mathbb{N}$,

$$\begin{aligned} |b_{2^{j-1}d}(x)-b_{2^jd}(x)| & \le \frac{1}{\mu(B(x,2^{j-1}d))}\int_{B(x,2^{j-1}d)} |b(y)-b_{2^jd}(x)|d\mu(y) \\ & \le C\frac{1}{\mu(B(x,2^jd))}\int_{B(x,2^jd)} |b(y)-b_{2^jd}(x)|d\mu(y) \\ & \le C\|b\|_{*,1}. \end{aligned} \tag{2.10}$$

Hence, it concludes that

$$|b_d(x)-b_{2^jd}(x)| \le Cj\|b\|_{*,1}, \quad j\in\mathbb{N},\ x\in X. \tag{2.11}$$

Then, since $(X,\delta,\mu)$ is normal, it follows

$$\begin{aligned} d^\gamma & \int_{X\setminus B(x,d)} \frac{|b(y)-b_d(x)|^{p'}}{\delta(x,y)^{1+\gamma}}d\mu(y) \\ & \le d^\gamma \sum_{j=0}^{\infty}\int_{2^{j+1}d>\delta(x,y)\ge 2^jd} \frac{|b(y)-b_d(x)|^{p'}}{\delta(x,y)^{1+\gamma}}d\mu(y) \\ & \le Cd^\gamma \sum_{j=0}^{\infty}(2^jd)^{-1-\gamma}\int_{2^{j+1}d>\delta(x,y)\ge 2^jd} |b(y)-b_d(x)|^{p'}d\mu(y) \\ & \le C\sum_{j=0}^{\infty}\frac{2^{-j\gamma}}{2^jd}\bigg(\int_{B(x,2^{j+1}d)} |b(y)-b_{2^{j+1}d}(x)|^{p'}d\mu(y) \\ & \qquad + \left((j+1)\|b\|_{*,1}\right)^{p'}\mu(B(x,2^{j+1}d))\bigg) \\ & \le C\sum_{j=0}^{\infty}2^{-j\gamma}\bigg(\frac{1}{\mu(B(x,2^{j+1}d))}\int_{B(x,2^{j+1}d)} |b(y)-b_{2^{j+1}d}(x)|^{p'}d\mu(y) \\ & \qquad + \left((j+1)\|b\|_{*,1}\right)^{p'}\bigg) \le \|b\|_{*,p'}^{p'}. \end{aligned} \tag{2.12}$$

On the other hand, if $\delta(x_0,y)\le\beta d$ and $\delta(x,y)\le d$, then $\delta(x_0,y)\ge((1-k\beta)/k)d$ and $\delta(x_0,y)\le k(\beta+1)\delta(x,y)$. Hence, by invoking again the normality of $(X,\delta,\mu)$ we can write

$$\begin{aligned}
d^{\gamma}\int_{X\setminus B(x,d)}&\frac{|f(y)|^p}{\delta(x,y)^{1+\gamma-p\alpha}}d\mu(y)\\
&\le Cd^{\gamma}\int_{X\setminus B(x_0,((1-k\beta)/k)d)}\frac{|f(y)|^p}{\delta(x_0,y)^{1+\gamma-p\alpha}}d\mu(y)\\
&\le Cd^{\gamma}\sum_{j=0}^{\infty}\int_{2^{j+1}((1-k\beta)/k)d>\delta(x_0,y)\ge 2^j((1-k\beta)/k)d}\frac{|f(y)|^p}{\delta(x_0,y)^{1+\gamma-p\alpha}}d\mu(y)\\
&\le Cd^{\gamma}\sum_{j=0}^{\infty}(d2^j)^{-1-\gamma+p\alpha}\int_{B(x_0,2^{j+1}((1-k\beta)/k)d)}|f(y)|^p d\mu(y)\\
&\le C\sum_{j=0}^{\infty}2^{-j\gamma}\frac{1}{\mu(B(x_0,2^{j+1}((1-k\beta)/k)d))^{1-p\alpha}}\int_{B(x_0,2^{j+1}((1-k\beta)/k)d)}|f(y)|^p d\mu(y)\\
&\le CM_{p\alpha}(|f|^p)(x_0).
\end{aligned}\tag{2.13}$$

Thus the result is proved. □

The following Whitney type covering lemma will be useful in the sequel.

**Lemma 2.5** (see [4, Lemma 1] and [13, Lemma 2]). *Let $\Omega$ be a proper open subset of $X$ and let $B$ be a ball in $X$ such that $B\cap\Omega\neq\emptyset$ and $B\cap(X\setminus\Omega)\neq\emptyset$. Then there exists a sequence $(B_j)_{j\in\mathbb{N}}$ of balls in $X$ satisfying the following three properties:*

(i) $\Omega\cap B\subset\cup_{j\in\mathbb{N}}B_j\subset\Omega\cap(B^*)^*$,

(ii) $B_j^*\cap(X\setminus\Omega)\neq\emptyset$, $j\in\mathbb{N}$, *and*

(iii) $\mu(\Omega\cap B)\le\sum_{j=1}^{\infty}\mu(B_j)\le C\mu(\Omega\cap(B^*)^*)$.

*Here if $B=B(x,r)$, with $x\in X$ and $r>0$, $B^*$ denotes the ball $B(x,rk(2k+1))$.*

Next we will prove in the main lemma a *good*-$\lambda$ inequality.

**Lemma 2.6.** *Let $0\le\rho<1$ and $1<p<1/\alpha$. Assume that $(X,\delta,\mu)$ is a normal space of homogeneous type that is of order $\rho$ and such that $\mu(\{x\})=0$, $x\in X$. Let $b\in BMO$ and $f$ be a measurable function on $X$. Then there exists $\beta_0$ such that for every $\beta\ge\beta_0$ and $\gamma>1$*

$$\begin{aligned}
\mu\Big(\Big\{x\in X:C(b,f)(x)>\beta\lambda,\ \|b\|_{*,p'}\Big(I_\alpha(|f|)(x)+\big(M_{p\alpha}(|f|^p)(x)\big)^{1/p}\Big)\le\gamma\lambda\Big\}\Big)&\\
\le C\gamma\mu(\{x\in X:C(b,f)(x)>\lambda\}),&
\end{aligned}\tag{2.14}$$

*provided that one of the following two conditions holds:*

(i) $\lambda>0$ *and* $\mu(X)=\infty$,

(ii) $\lambda>(C_0/\mu(X))^{1-\alpha}\|b\|_{p'}\|f\|_p$ *and* $\mu(X)<\infty$, *where $C_0$ is the positive constant appearing in Lemma 2.3.*

**Proof.** Let $\beta$, $\gamma>0$ and $\lambda$ satisfying the imposed conditions. We define the following sets:

$$\begin{aligned}
E_\lambda(\beta,\gamma)&=\Big\{x\in X:C(b,f)(x)>\beta\gamma,\\
&\qquad\|b\|_{*,p'}\Big(I_\alpha(|f|)(x)+\big(M_{p\alpha}(|f|^p)(x)\big)^{1/p}\Big)\le\gamma\lambda\Big\},\\
W_\lambda&=\{x\in X:C(b,f)(x)>\lambda\}.
\end{aligned}\tag{2.15}$$

Note that we can assume, without loss of generality, that $W_\lambda \neq \emptyset$ and $W_\lambda \neq X$. Indeed, suppose firstly that $\mu(X) = \infty$. If $W_\lambda = X$, then (2.14) is clear for every $\beta > 0$ and $\gamma > 0$. On the other hand, if $\mu(X) < \infty$, then Lemma 2.3 implies that $\mu(W_\lambda) < \mu(X)$ when $\lambda > (C_0/\mu(X))^{1-\alpha}\|b\|_{p'}\|f\|_p$ and where $C_0$ is the positive constant that appears in Lemma 2.3. Also if $\mu(X) \leq \infty$ and $W_\lambda = \emptyset$, then (2.14) holds for every $\beta > 1$ and $\gamma > 0$.

Let $B$ be a ball in X such that $B \cap W_\lambda \neq \emptyset$ and $B \cap (X \setminus W_\lambda) \neq \emptyset$. Then there exists a sequence $(B_j)_{j=1}^{\infty}$ of balls in $X$ satisfying conditions (i), (ii), and (iii) in Lemma 2.5 by replacing $\Omega$ by $W_\lambda$.

Let $j \in \mathbb{N}$. Write $B_j = B(a,d)$, with $a \in X$ and $d > 0$. We define $B_j^1 = B(a,\alpha_1 d)$ and $B_j^2 = B(a,\alpha_2 d)$, where $\alpha_1 \leq k(2k^2(1+k(2k+1))+1)$ and $\alpha_2 > k(1+k(\alpha_1+1))$.

Assume that $B_j \cap E_\lambda(\beta,\gamma) \neq \emptyset$ and choose $x_1 \in B_j \cap E_\lambda(\beta,\gamma)$. We write $f = f_1 + f_2$, where $f_1 = f\chi_{B_j^1}$, and $b = b_1 + b_2$, being $b_1 = (b - b_{B_j^2})\chi_{B_j^2}$ and $b_{B_j^2} = 1/\mu(B_j^2) \times \int_{B_j^2} b(y)\,d\mu(y)$.

We have that $C(b,f_1)(x) \leq C(b_1,f_1)(x)$, for every $x \in B_j$. Indeed, let $x \in B_j$ and $\epsilon > 0$. Since $\alpha_2 > k(1+k(\alpha_1+1))$, if $B(x,\epsilon) \cap (X \setminus B_j^2) \neq \emptyset$, then $B_j^1 \subset B(x,\epsilon)$. Hence we can write

$$\begin{aligned}(b_1)_\epsilon(x) &= \frac{1}{\mu(B(x,\epsilon))}\int_{B(x,\epsilon)} b_1(y)\,d\mu(y)\\ &= \frac{1}{\mu(B(x,\epsilon))}\int_{B(x,\epsilon)\cap B_j^2}\Big(b(y) - b_{B_j^2}\Big)d\mu(y)\\ &= \frac{1}{\mu(B(x,\epsilon))}\int_{B(x,\epsilon)} b(y)\,d\mu(y) - b_{B_j^2} = b_\epsilon(x) - b_{B_j^2},\end{aligned} \tag{2.16}$$

provided that $B_j^1 \cap (X \setminus B(x,\epsilon)) \neq \emptyset$.

Then, since $B_j^1 \subset B_j^2$, one has

$$\begin{aligned}C(b,f_1)(x) &= \sup_{\epsilon>0}\left|\int_{X\setminus B(x,\epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_1(y)\,d\mu(y)\right|\\ &= \sup_{\epsilon>0}\left|\int_{(X\setminus B(x,\epsilon))\cap B_j^1} \frac{b_1(y) + b_{B_j^2} - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_1(y)\,d\mu(y)\right|\\ &\leq \sup_{\epsilon>0}\left|\int_{X\setminus B(x,\epsilon)} \frac{b_1(y) - (b_1)_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_1(y)\,d\mu(y)\right| = C(b_1,f_1)(x).\end{aligned} \tag{2.17}$$

Moreover from Lemma 2.3 we deduce that for every $\beta > 1$,

$$\begin{aligned}&\mu(\{x \in B_j : C(b_1,f_1)(x) > \beta\lambda\})\\ &\leq C\left(\frac{\|b_1\|_{p'}\|f_1\|_p}{\beta\lambda}\right)^{1/1-\alpha}\\ &= C\left(\int_{B_j^2}\left|b(y) - b_{B_j^2}\right|^{p'} d\mu(y)\right)^{1/p'(1-\alpha)}\left(\int_{B_j^1}|f(y)|^p\,d\mu(y)\right)^{1/p(1-\alpha)}\\ &\leq C\lambda^{1/\alpha-1}\mu(B_j)\Big(\|b\|_{*,p'}(M_{p\alpha}(|f|^p)(x_1))^{1/p}\Big)^{1/1-\alpha},\end{aligned} \tag{2.18}$$

because $\mu$ is doubling.

Hence, since $x_1 \in B_j \cap E_\lambda(\beta,\gamma)$ if $\gamma < 1$, then

$$\mu(\{x \in B_j : C(b,f_1)(x) > \beta\lambda\}) \le C\gamma\mu(B_j). \tag{2.19}$$

By virtue of (ii) in Lemma 2.5, $B_j^* \cap (X \setminus W_\lambda) \neq \emptyset$. Choose $x_0 \in B_j^* \cap (X \setminus W_\lambda)$, that is, $x_0 \in B_j^*$ and $C(b,f)(x_0) \le \lambda$.

Now our objective is to estimate

$$\mu(\{x \in B_j : C(b,f_2)(x) > \beta\lambda\}). \tag{2.20}$$

We consider two cases.

Assume firstly that $\epsilon > \sigma d$, where $\alpha_2/k - 1 > \sigma > (\alpha_1+1)k$. Since $\sigma > (\alpha_1+1)k$, for every $x \in B_j$, $B_j^1 \subset B(x,\epsilon)$. Let $x \in B_j$. We have

$$\begin{aligned}\int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f_2(y)\,d\mu(y) &= \int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f(y)\,d\mu(y) \\ &= I_1 + I_2 + I_3,\end{aligned} \tag{2.21}$$

where

$$\begin{aligned} I_1 &= \int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f(y)\,d\mu(y) - \int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_\epsilon(x_0)}{\delta(x,y)^{1-\alpha}} f(y)\,d\mu(y), \\ I_2 &= \int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_\epsilon(x_0)}{\delta(x,y)^{1-\alpha}} f(y)\,d\mu(y) - \int_{X\setminus B(x_0,\epsilon)} \frac{b(y)-b_\epsilon(x_0)}{\delta(x_0,y)^{1-\alpha}} f(y)\,d\mu(y), \\ I_3 &= \int_{X\setminus B(x_0,\epsilon)} \frac{b(y)-b_\epsilon(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y)\,d\mu(y). \end{aligned} \tag{2.22}$$

We are going to estimate $I_i$, $i = 1,2,3$.

As mentioned above if $\delta(x,y) > \epsilon$, then $y \notin B_j^1$. Hence $\delta(x,y) > \epsilon$ implies that $\delta(x,y) \ge ((\alpha_1 - k)/k)d > 0$. Then we can write

$$\frac{\delta(x_1,y)}{\delta(x,y)} \le \frac{k(\delta(x,x_1)+\delta(x,y))}{\delta(x,y)} \le \frac{2k^3}{\alpha_1 - k} + k, \tag{2.23}$$

provided that $\delta(x,y) > \epsilon$.

Therefore it follows

$$\begin{aligned} |I_1| &= \left| \int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f(y)\,d\mu(y) - \int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_\epsilon(x_0)}{\delta(x,y)^{1-\alpha}} f(y)\,d\mu(y) \right| \\ &\le \int_{X\setminus B(x,\epsilon)} \frac{|b_\epsilon(x_0)-b_\epsilon(x)|}{\delta(x,y)^{1-\alpha}} |f(y)|\,d\mu(y) \\ &\le C|b_\epsilon(x_0)-b_\epsilon(x)| \int_{X\setminus B_j^1} \frac{|f(y)|}{\delta(x_1,y)^{1-\alpha}}\,d\mu(y) \\ &\le C|b_\epsilon(x_0)-b_\epsilon(x)|\, I_\alpha(|f|)(x_1). \end{aligned} \tag{2.24}$$

Moreover if $y \in B(x_0,\epsilon)$, then

$$\delta(x,y) \le k(\delta(y,x_0)+\delta(x_0,x)) \le k(\epsilon + kd(1+k(2k+1))) < 2^m\epsilon, \tag{2.25}$$

where $m \in \mathbb{N}$ is large enough and $m$ is not depending on $d$ and $\epsilon$.

Hence, since $(X,\delta,\mu)$ is normal we have that

$$\begin{aligned}
&|b_\epsilon(x_0) - b_\epsilon(x)| \\
&\quad \le \frac{1}{\mu(B(x_0,\epsilon))}\int_{B(x_0,\epsilon)} |b(y) - b_\epsilon(x)|d\mu(y) \\
&\quad \le C\frac{1}{\mu(B(x,2^m\epsilon))}\int_{B(x,2^m\epsilon)} |b(y) - b_\epsilon(x)|d\mu(y) \\
&\quad \le C\left(\frac{1}{\mu(B(x,2^m\epsilon))}\int_{B(x,2^m\epsilon)} |b(y) - b_{2^m\epsilon}(x)|d\mu(y) + |b_{2^m\epsilon}(x) - b_\epsilon(x)|\right) \\
&\quad \le C\|b\|_{*,p'}.
\end{aligned} \tag{2.26}$$

Thus we conclude that

$$|I_1| \le C\|b\|_{*,p'} I_\alpha(|f|)(x_1) \le C\gamma\lambda. \tag{2.27}$$

On the other hand, to estimate $I_2$ we will use that $(X,\delta,\mu)$ is a space of homogeneous type which is of order $\rho \in (0,1)$. It is clear that

$$\begin{aligned}
|I_2| \le &\int_{\delta(x,y)\ge\epsilon \text{ and } \delta(x_0,y)\ge\epsilon} |b(y) - b_\epsilon(x_0)||f_2(y)|\left|\delta(x,y)^{\alpha-1} - \delta(x_0,y)^{\alpha-1}\right|d\mu(y) \\
&+ \left|\int_{\delta(x,y)\ge\epsilon \text{ and } \delta(x_0,y)<\epsilon} \frac{b(y) - b_\epsilon(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y)d\mu(y)\right. \\
&\quad \left. - \int_{\delta(x_0,y)\ge\epsilon \text{ and } \delta(x,y)<\epsilon} \frac{b(y) - b_\epsilon(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y)d\mu(y)\right|.
\end{aligned} \tag{2.28}$$

Note that, since $\sigma > 2k^2(1+k(2k+1))$, $\delta(x_0,y) \le 2k\delta(x_0,x)$ provided that $\delta(x_0,y) > \epsilon$. Hence, according to [11, Lemma II.3] and Lemma 2.4, since $\delta(x_0,x_1) < (1/2k)\epsilon$, we have,

$$\begin{aligned}
&\int_{\delta(x,y)\ge\epsilon \text{ and } \delta(x_0,y)\ge\epsilon} |b(y) - b_\epsilon(x_0)||f_2(y)|\left|\delta(x,y)^{\alpha-1} - \delta(x_0,y)^{\alpha-1}\right|d\mu(y) \\
&\qquad \le C\delta(x,x_0)^\rho \int_{X\setminus B(x_0,\epsilon)} |b(y) - b_\epsilon(x_0)||f_2(y)|\left|\delta(x_0,y)^{\alpha-\rho-1}\right|d\mu(y) \\
&\qquad \le C\epsilon^\rho \int_{X\setminus B(x_0,\epsilon)} |b(y) - b_\epsilon(x_0)||f_2(y)|\left|\delta(x_0,y)^{\alpha-\rho-1}\right|d\mu(y) \\
&\qquad \le C\|b\|_{*,p'}\left(M_{p\alpha}(|f|^p)(x_1)\right)^{1/p} \le C\gamma\lambda.
\end{aligned} \tag{2.29}$$

Moreover, $\delta(x,y) < \epsilon$ implies that $\delta(x_0,y) \le \epsilon(k+(1/2))$ and this inequality implies that $\delta(x_1,y) \le \epsilon(k(k+(1/2))+(1/2))$. Then, by taking into account the normality of

$(X,\delta,\mu)$, Holder inequality leads to

$$\begin{aligned}
&\bigg|\int_{\delta(x,y)\geq\epsilon \text{ and } \delta(x_0,y)<\epsilon}\frac{b(y)-b_\epsilon(x_0)}{\delta(x,y)^{1-\alpha}}f_2(y)\,d\mu(y)\\
&\quad-\int_{\delta(x_0,y)\geq\epsilon \text{ and } \delta(x,y)<\epsilon}\frac{b(y)-b_\epsilon(x_0)}{\delta(x_0,y)^{1-\alpha}}f_2(y)\,d\mu(y)\bigg|\\
&\leq C\epsilon^{\alpha-1}\int_{B(x_0,\epsilon(k+(1/2)))}\left|b(y)-b_\epsilon(x_0)\right|\left|f_2(y)\right|d\mu(y)\\
&\leq C\frac{1}{\mu(B(x_0,\epsilon(k+(1/2))))^{1-\alpha}}\int_{B(x_0,\epsilon(k+(1/2)))}\left|b(y)-b_\epsilon(x_0)\right|\left|f_2(y)\right|d\mu(y)\\
&\leq \|b\|_{*,p'}\left(M_{p\alpha}(|f|^p)(x_1)\right)^{1/p}\leq C\gamma\lambda.
\end{aligned} \tag{2.30}$$

Finally, since $x_0\notin W_\lambda$, we have

$$\left|I_3\right|\leq\lambda. \tag{2.31}$$

By combining (2.21), (2.27), and (2.31) we conclude that

$$\sup_{\epsilon>d\sigma}\left|\int_{X\setminus B(x,\epsilon)}\frac{b(y)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}}f_2(y)\,d\mu(y)\right|\leq C\gamma\lambda+\lambda. \tag{2.32}$$

Assume now $0<\epsilon\leq d\sigma$. Let $x\in B_j$. It is not hard to see that if $y$ is in the support of $f_2$ then $\delta(x,y)\geq((\alpha_1-k)/k)d$ and $\delta(x_0,y)\geq((\alpha_1-k^2(2k+1))/k)d$. We choose $\omega>0$ such that $\omega<(\alpha_1-k)/k$.

We can write

$$\int_{X\setminus B(x,\epsilon)}\frac{b(y)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}}f_2(y)\,d\mu(y)=J_1+J_2+J_3, \tag{2.33}$$

where

$$\begin{aligned}
J_1&=\int_{X\setminus B(x,\epsilon)}\frac{b_{\omega d}(x_0)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}}f_2(y)\,d\mu(y),\\
J_2&=\int_{X\setminus B(x,\epsilon)}\frac{b(y)-b_{\omega d}(x_0)}{\delta(x,y)^{1-\alpha}}f_2(y)\,d\mu(y)-\int_{X\setminus B(x_0,\epsilon)}\frac{b(y)-b_{\omega d}(x_0)}{\delta(x_0,y)^{1-\alpha}}f_2(y)\,d\mu(y),\\
J_3&=\int_{X\setminus B(x_0,\epsilon)}\frac{b(y)-b_{\omega d}(x_0)}{\delta(x_0,y)^{1-\alpha}}f_2(y)\,d\mu(y).
\end{aligned} \tag{2.34}$$

We will estimate $J_i$, $i=1,2,3$.

By proceeding as in the study of $I_1$, since $k(\sigma+1)<\alpha_2$, we obtain

$$\begin{aligned}
\left|J_1\right|&\leq C\left|b_{\omega d}(x_0)-b_\epsilon(x)\right|I_\alpha(|f|)(x_1)\\
&\leq C\frac{1}{\mu(B(x,\epsilon))}\int_{B(x,\epsilon)}\left|b(y)-b_{\omega d}(x_0)\right|\chi_{B_j^2}(y)\,d\mu(y)I_\alpha(|f|)(x_1)\\
&\leq CM\left((b-b_{\omega d}(x_0))\chi_{B_j^2}\right)(x)I_\alpha(|f|)(x_1).
\end{aligned} \tag{2.35}$$

On the other hand, we have that

$$
\begin{aligned}
J_2 &= \int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_{\omega d}(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y)\,d\mu(y) - \int_{X\setminus B(x_0,\epsilon)} \frac{b(y)-b_{\omega d}(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y)\,d\mu(y) \\
&= \int_{\delta(x,y)\geq\epsilon \text{ and } \delta(x_0,y)\geq\epsilon} \left(b(y)-b_{\omega d}(x_0)\right) f_2(y)\left(\delta(x,y)^{\alpha-1}-\delta(x_0,y)^{\alpha-1}\right) d\mu(y) \\
&\quad + \int_{\delta(x,y)\geq\epsilon \text{ and } \delta(x_0,y)<\epsilon} \frac{b(y)-b_{\omega d}(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y)\,d\mu(y) \\
&\quad - \int_{\delta(x,y)<\epsilon \text{ and } \delta(x_0,y)\geq\epsilon} \frac{b(y)-b_{\omega d}(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y)\,d\mu(y).
\end{aligned}
\tag{2.36}
$$

Since $(X,\delta,\mu)$ is a space of homogeneous type of order $\rho\in(0,1)$, by virtue of [11, Lemma 2.3], we have

$$
\begin{aligned}
&\left|\int_{\delta(x,y)\geq\epsilon \text{ and } \delta(x_0,y)\geq\epsilon} \left(b(y)-b_{\omega d}(x_0)\right) f_2(y)\left(\delta(x,y)^{\alpha-1}-\delta(x_0,y)^{\alpha-1}\right) d\mu(y)\right| \\
&\quad \leq C\delta(x,x_0)^{\rho} \int_{\delta(x,y)\geq\epsilon \text{ and } \delta(x_0,y)\geq\epsilon} \left|b(y)-b_{\omega d}(x_0)\right| \left|f_2(y)\right| \delta(x,y)^{\alpha-\rho-1}\,d\mu(y),
\end{aligned}
\tag{2.37}
$$

because if $y$ is in the support of $f_2$, then $\delta(x,y)\geq((\alpha_1-k)/k)d\geq 2k^2(k(2k+1)+1)d\geq 2k\delta(x_0,x)$. Hence, since $y\in\operatorname{supp} f_2$ implies that $\delta(x_1,y)>\omega d$, by proceeding as in the proof of Lemma 2.4 we conclude

$$
\begin{aligned}
&\left|\int_{\delta(x,y)\geq\epsilon \text{ and } \delta(x_0,y)\geq\epsilon} \left(b(y)-b_{\omega d}(x_0)\right) f_2(y)\left(\delta(x,y)^{\alpha-1}-\delta(x_0,y)^{\alpha-1}\right) d\mu(y)\right| \\
&\qquad \leq C\delta(x,x_0)^{\rho} \int_{\delta(x_1,y)>\omega d} \frac{\left|b(y)-b_{\omega d}(x_0)\right|}{\delta(x_1,y)^{1+\rho-\alpha}} |f(y)|\,d\mu(y) \\
&\qquad \leq C\|b\|_{*,p'}\left(M_{p\alpha}(|f|^p)(x_1)\right)^{1/p} \leq C\gamma\lambda.
\end{aligned}
\tag{2.38}
$$

Also, since if $\delta(x,y)<\epsilon$, then $\delta(x_0,y)<dk(k(k(2k+1)+1)+\sigma)$ and since $\omega<\alpha_1$, we have

$$
\begin{aligned}
&\left|\int_{\delta(x,y)\geq\epsilon \text{ and } \delta(x_0,y)<\epsilon} \frac{b(y)-b_{\omega d}(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y)\,d\mu(y) \right. \\
&\qquad \left. - \int_{\delta(x,y)<\epsilon \text{ and } \delta(x_0,y)\geq\epsilon} \frac{b(y)-b_{\omega d}(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y)\,d\mu(y)\right| \\
&\quad \leq \int_{\omega d<\delta(x_0,y)<k(k(k(2k+1)+1)+\sigma)d} \left|b(y)-b_{\omega d}(x_0)\right| \left|f_2(y)\right| \\
&\qquad \times \left(\delta(x,y)^{\alpha-1}+\delta(x_0,y)^{\alpha-1}\right) d\mu(y) \\
&\quad \leq C\int_{\omega d<\delta(x_0,y)<k(k(k(2k+1)+1)+\sigma)d} \frac{\left|b(y)-b_{\omega d}(x_0)\right|}{\delta(x_0,y)^{1-\alpha}} \left|f_2(y)\right| d\mu(y).
\end{aligned}
\tag{2.39}
$$

Now by proceeding as in the proof of Lemma 2.4 we obtain that

$$\begin{aligned}
&\Big| \int_{\delta(x,y)\geq\epsilon \text{ and } \delta(x_0,y)<\epsilon} \frac{b(y)-b_{\omega d}(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y)\,d\mu(y) \\
&\quad - \int_{\delta(x,y)<\epsilon \text{ and } \delta(x_0,y)\geq\epsilon} \frac{b(y)-b_{\omega d}(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y)\,d\mu(y) \Big| \\
&\leq C\|b\|_{*,p'} \left(M_{p\alpha}(|f|^p)(x_1)\right)^{1/p} \leq C\gamma\lambda.
\end{aligned} \tag{2.40}$$

In a similar way we can see that

$$\begin{aligned}
|J_3| &\leq \int_{X\setminus B(x_0,\omega d)} \frac{|b(y)-b_{\omega d}(x_0)|}{\delta(x_0,y)^{1-\alpha}} |f_2(y)|\,d\mu(y) \\
&\leq C\|b\|_{*,p'} \left(M_{p\alpha}(|f|^p)(x_1)\right)^{1/p} \leq C\gamma\lambda.
\end{aligned} \tag{2.41}$$

By combining the above estimates we can conclude

$$\begin{aligned}
&\sup_{0<\epsilon\leq d\sigma} \Big| \int_{X\setminus B(x,\epsilon)} \frac{b(y)-b_{\epsilon}(x)}{\delta(x,y)^{1-\alpha}} f_2(y)\,d\mu(y) \Big| \\
&\qquad \leq C\Big(\lambda\gamma + I_\alpha(|f|)(x_1) M\Big((b-b_{\omega d}(x_0))\chi_{B_j^2}\Big)(x)\Big).
\end{aligned} \tag{2.42}$$

From (2.32) and (2.42) follows that for every $x \in B_j$

$$C(b,f_2)(x) \leq C\Big(\lambda\gamma + \lambda + I_\alpha(|f|)(x_1) M\Big((b-b_{\omega d}(x_0))\chi_{B_j^2}\Big)(x)\Big). \tag{2.43}$$

Hence if $\beta$ is large enough, then according to Lemma 2.2 and since $\mu$ is doubling

$$\begin{aligned}
&\mu(\{x \in B_j : C(b,f_2)(x) > \beta\lambda\}) \\
&\quad \leq \mu\Big(\Big\{x \in B_j : I_\alpha(|f|)(x_1) M\Big((b-b_{\omega d}(x_0))\chi_{B_j^2}\Big)(x) > \lambda\Big\}\Big) \\
&\quad \leq C\lambda^{-1} I_\alpha(|f|)(x_1) \int_{B_j^2} |b(y)-b_{\omega d}(x_0)|\,d\mu(y) \\
&\quad \leq C\lambda^{-1} I_\alpha(|f|)(x_1) \|b\|_{*,p'} \mu(B_j) \leq C\gamma\mu(B_j).
\end{aligned} \tag{2.44}$$

Thus we obtain that for $\beta \geq \beta_0$ and $\gamma < 1$, where $\beta_0$ is large enough,

$$\mu(B_j \cap E_\lambda(\beta,\gamma)) \leq C\gamma\mu(B_j). \tag{2.45}$$

Hence

$$\mu(B \cap E_\lambda(\beta,\gamma)) \leq C\gamma \sum_{j=1}^{\infty} \mu(B_j) \leq C\gamma\mu(W_\lambda), \quad \beta \geq \beta_0,\ \gamma < 1. \tag{2.46}$$

Arbitrariness of $B$ allows to conclude that

$$\mu(E_\lambda(\beta,\gamma)) \leq C\gamma\mu(W_\lambda), \quad \beta \geq \beta_0,\ \gamma < 1, \tag{2.47}$$

and the proof is finished. □

**Proof of Theorem 1.1.** To prove Theorem 1.1 we proceed as in the proof of [6, Theorem III]. We start proving that the operator $C(b,f)$ is bounded from $L^p(X,\mu)$ into $L^q(X,\mu)$, when $1<p<1/\alpha$ and $1/q=1/p-\alpha$. Assume that $b\in L^\infty(X,\mu)$.

Let $1<p_1<p<1/\alpha$ and $1/q=1/p-\alpha$. Assume firstly that $\mu(X)=\infty$. According to Lemma 2.6, $f\in L^p(X,\mu)$ we have

$$\begin{aligned}
&\int_X (C(b,f)(x))^q d\mu(x)\\
&= \beta^q q\int_0^\infty \lambda^{q-1}\mu(\{x\in X : C(b,f)(x)>\beta\lambda\})d\lambda\\
&\le C\beta^q\Big(\gamma\int_0^\infty \lambda^{q-1}\mu(\{x\in X : C(b,f)(x)>\lambda\})d\lambda\\
&\qquad +\int_0^\infty \lambda^{q-1}\mu\Big(\Big\{x\in X : \|b\|_{*,p'}\Big(I_\alpha(|f|)(x)+(M_{p_1\alpha}(|f|^{p_1})(x))^{1/p_1}\Big)>\gamma\lambda\Big\}\Big)d\lambda\Big)\\
&= C\beta^q\Big(\gamma\int_X (C(b,f)(x))^q d\mu(x)\\
&\qquad +\gamma^{-q}\|b\|_{*,p'}^q\int_X\Big(I_\alpha(|f|)(x)+(M_{p_1\alpha}(|f|^{p_1})(x))^{1/p_1}\Big)^q d\mu(x)\Big),
\end{aligned}\tag{2.48}$$

provided that $\beta\ge\beta_0$ and $0<\gamma<1$, where $\beta_0$ is given in Lemma 2.6.

Hence by (2.4) and Lemma 2.1 and by taking $\gamma$ so small we can conclude that

$$\|C(b,f)\|_q\le C\|b\|_{*,p'}\Big(\|I_\alpha(|f|)\|_q+\|M_{p_1\alpha}(|f|^{p_1})\|_{q/p_1}^{1/p_1}\Big).\tag{2.49}$$

According to Lemmas 2.1 and 2.2 it follows

$$\|C(b,f)\|_q\le C\|b\|_{*,p'}\|f\|_p.\tag{2.50}$$

Suppose now that $\mu(X)<\infty$. Since $C(b,f)=C(b-a,f)$, for every $a\in\mathbb{C}$, we can assume, without loss of generality, that $\int_X b\,d\mu=0$. Then Lemma 2.6 leads, for every $f\in L^p(X,\mu)$, to

$$\begin{aligned}
&\int_X (C(b,f)(x))^q d\mu(x)\\
&= \beta^q q\int_0^\infty \lambda^{q-1}\mu(\{x\in X : C(b,f)(x)>\beta\lambda\})d\lambda\\
&\le C\beta^q\Big(\int_0^{\|b\|_{p'}\|f\|_p(C_0/\mu(X))^{1-\alpha}}\lambda^{q-1}d\lambda+\gamma\int_X (C(b,f)(x))^q d\mu(x)\\
&\qquad +\gamma^{-q}\|b\|_{*,p'}^q\int_X\Big(I_\alpha(|f|)(x)+(M_{p_1\alpha}(|f|^{p_1})(x))^{1/p_1}\Big)^q d\mu(x)\Big)\\
&\le C\beta^q\Big(\|b\|_{p'}^q\|f\|_p^q+\gamma\int_X (C(b,f)(x))^q d\mu(x)\\
&\qquad +\gamma^{-q}\|b\|_{*,p'}^q\int_X\Big(I_\alpha(|f|)(x)+(M_{p_1\alpha}(|f|^{p_1})(x))^{1/p_1}\Big)^q d\mu(x)\Big),
\end{aligned}\tag{2.51}$$

when $\beta\ge\beta_0$ and $0<\gamma<1$, $\beta_0$ being as in Lemma 2.6.

Thus we deduce from Lemmas 2.1 and 2.2 that

$$\|C(b,f)\|_q\le C\|b\|_{*,p'}\|f\|_p.\tag{2.52}$$

Now we note that

$$
\begin{aligned}
[b,I_\alpha](f)(x) &= \lim_{\epsilon\to 0^+}\left(b(x)\int_{X\setminus B(x,\epsilon)}\frac{f(y)}{\delta(x,y)^{1-\alpha}}d\mu(y)-\int_{X\setminus B(x,\epsilon)}\frac{b(y)f(y)}{\delta(x,y)^{1-\alpha}}d\mu(y)\right)\\
&= \lim_{\epsilon\to 0^+}\left((b(x)-b_\epsilon(x))\int_{X\setminus B(x,\epsilon)}\frac{f(y)}{\delta(x,y)^{1-\alpha}}d\mu(y)\right.\\
&\qquad\left.-\int_{X\setminus B(x,\epsilon)}\frac{b(y)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}}f(y)d\mu(y)\right)\\
&= -\lim_{\epsilon\to 0^+}\int_{X\setminus B(x,\epsilon)}\frac{b(y)-b_\epsilon(x)}{\delta(x,y)^{1-\alpha}}f(y)d\mu(y),
\end{aligned}
\tag{2.53}
$$

for every $f\in L^p(X,\mu)$, and a.e. $x\in X$.

Then

$$
\|[b,I_\alpha]\|_q \le \|C(b,f)\|_q, \tag{2.54}
$$

for each $f\in L^p(X,\mu)$.

To finish the proof it is sufficient to take into account [3, Lemma 2.5] and Fatou's lemma. □

**Acknowledgement.** The research was partially supported by DGICYT Grant PB 94-0591 (Spain).

## References


[1] A. Bernardis and O. Salinas, *Two-weight norm inequalities for the fractional maximal operator on spaces of homogeneous type*, Studia Math. **108** (1994), no. 3, 201–207. MR 94m:42038. Zbl 846.42013.

[2] M. Bramanti, *Commutators of integral operators with positive kernels*, Matematiche (Catania) **49** (1994), no. 1, 149–168 (1995). MR 97c:47059. Zbl 840.42009.

[3] M. Bramanti and M. C. Cerutti, *Commutators of singular integrals and fractional integrals on homogeneous spaces*, Harmonic Analysis and Operator Theory (Caracas, 1994) (Providence, RI), Contemp. Math., 189, Amer. Math. Soc., 1995, pp. 81–94. MR 96m:42024. Zbl 837.43013.

[4] N. Burger, *Espace des fonctions a variation moyenne bornee sur un espace de nature homogene*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 3, A139–A142. MR 57#7041. Zbl 368.46037.

[5] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), no. 1, 7–16. MR 84j:42027. Zbl 523.42015.

[6] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250. MR 50#10670. Zbl 291.44007.

[7] R. R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), no. 3, 611–635. MR 54#843. Zbl 326.32011.

[8] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes*, Lecture Notes in Mathematics, Vol. 242, Springer-Verlag, Berlin, New York, 1971. MR 58#17690. Zbl 224.43006.

[9] ———, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645. MR 56#6264. Zbl 358.30023.

[10] A. E. Gatto, C. Segovia, and S. Vagi, *On fractional differentiation and integration on spaces of homogeneous type*, Rev. Mat. Iberoamericana **12** (1996), no. 1, 111–145. MR 97k:42028. Zbl 921.43005.

[11] A. E. Gatto and S. Vagi, *Fractional integrals on spaces of homogeneous type*, Analysis and Partial Differential Equations (New York), Dekker, 1990, pp. 171–216. MR 91e:42032. Zbl 695.43006.

[12] V. M. Kokilashvili and A. Kufner, *Fractional integrals on spaces of homogeneous type*, Comment. Math. Univ. Carolin. **30** (1989), no. 3, 511–523. MR 91e:42027. Zbl 686.42013.

[13] R. L. Long and L. Yang, *BMO functions in spaces of homogeneous type*, Sci. Sinica Ser. A **27** (1984), no. 7, 695–708. MR 85m:42022. Zbl 572.42011.

[14] J. Stromberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Mathematics, 1381, Springer-Verlag, Berlin, 1989. MR 90j:42053. Zbl 676.42021.

[15] R. L. Wheeden, *A characterization of some weighted norm inequalities for the fractional maximal function*, Studia Math. **107** (1993), no. 3, 257–272. MR 94m:42044. Zbl 809.42009.

Jorge J. Betancor: Departamento de Análisis Matemático, Universidad de La Laguna, 38271-La Laguna, Tenerife Islas Canarias, España

*E-mail address*: jbetanco@ull.es