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MULTIMODAL CYCLES WITH LINEAR MAP HAVING EXACTLY ONE FIXED POINT

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ABSTRACT. We describe a class of cycles that cannot be forced by a cycle whose linear map has exactly one fixed point.

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1. Introduction. This note is concerned with the forcing relation on cycles. In particular, we consider cycles θ for which the θ-linear map has exactly one fixed point. We prove a theorem which describes a large class of cycles that cannot be forced by θ.

2. Definitions. Throughout this note, $f : I \to I$ denotes a continuous map of a compact interval. For $x \in I$, $f^0(x) = x$, and for $n \in N$, $f^n(x) = f(f^{n-1}(x))$. An element $x \in I$ is a periodic point for f if there exists $k \in N$ satisfying $f^k(x) = x$. The least such k is called the period of x. A point of period 1 is called a fixed point. The orbit of $x \in I$ is the set $\{f^n(x)\}_{n=0}^{\infty}$ and is denoted $\mathcal{O}(x)$. If x is periodic with period k, then $\mathcal{O}(x)$ is a finite set consisting of k distinct elements.

A cycle of order n is a bijection $\theta : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ satisfying $\theta^k(1) \neq 1$ for $1 \le k < n$. Let x be a periodic point for f with least period n and $\mathcal{O}(x) = \{x_1 < x_2 < \cdots < x_n\}$. We say that x has orbit type θ if θ is a cycle of order n and $f(x_i) = x_{\theta(i)}$ for $1 \le i \le n$. In this case, we also say that the periodic orbit $\mathcal{O}(x)$ has orbit type θ. We say that f has a periodic orbit of orbit type θ if there exists a periodic point $x \in I$ which has orbit type θ. A cycle θ forces a cycle η if whenever f has a periodic orbit of type θ, f has a periodic point of type η.

For a cycle θ of order n, the θ-linear map $L_\theta : [1,n] \to [1,n]$ is defined by

$$\begin{gathered} L_\theta(k) = \theta(k), \quad \text{for } 1 \le k \le n, \\ L_\theta \text{ is linear on } [i,i+1], \quad \text{for } 1 \le i \le n-1. \end{gathered} \tag{2.1}$$

The graph of L_θ consists of at most $n-1$ linear segments, each having a slope m satisfying $|m| \ge 1$. A cycle η is forced by θ if and only if L_θ has a periodic orbit of type η [1].

Baldwin [2] defined the forcing relation and proved that the forcing relation induces a partial order on the set of cycles. He provided an exhaustive but inefficient algorithm for determining whether one cycle forces another. Jungreis [6] provided a combinatorial method to determine if one cycle forces another in certain cases. In [3] a geometric version of Jungreis's algorithm is given and in [4] this algorithm is generalized to any

two cycles. In [8], another geometric algorithm is given to determine the forcing relation. This algorithm is similar to Baldwin's original algorithm but more efficient. A cycle is called unimodal if L_θ has exactly one turning point (a maximum, say). In [5] the forcing relation on the set of unimodal cycles is studied. In particular, it is shown that the forcing relation induces a total order on the set of unimodal cycles. In [7, 9] the structure of this totally ordered set is investigated.

3. Preliminaries. In this section, we define the RL-pattern for any cycle, and we define the step number for a cycle θ for which L_θ has exactly one fixed point.

Definition 3.1. Let η be any cycle of order k. The RL-pattern for η is the sequence

$$G = G_1 G_2 \cdots G_k \in \{R, L\}^k \tag{3.1}$$

defined by

$$G_i = \begin{cases} R & \text{if } \eta^i(1) > \eta^{i-1}(1), \\ L & \text{if } \eta^i(1) < \eta^{i-1}(1). \end{cases} \tag{3.2}$$

Let $R(\eta)$ denote the length of the longest string of consecutive R's in the RL-pattern for η.

Obviously, every RL-pattern begins with an R and ends with an L.

Let θ be a cycle of order n such that L_θ has exactly one fixed point. Let $p_1 \in (1, n)$ denote the unique fixed point and let $E_1 = \{x < p_1 \mid f(x) = p_1\}$. If $E_1 \neq \varnothing$, we let $p_2 = \max\{E_1\}$. For $i > 1$, if the points $p_1, p_2, \ldots, p_i$ and nonempty sets $E_1, \ldots, E_{i-1}$ have been defined, we set

$$E_i = \{x < p_i \mid f(x) = p_i\}. \tag{3.3}$$

If $E_i \neq \varnothing$, we let $p_{i+1} = \max\{E_i\}$. We see that for some $i \geq 1$, $E_i = \varnothing$, for otherwise, there would exist a strictly decreasing sequence $\{p_n\}_{n=1}^{\infty}$ in $[1, n]$, converging to a point $p < p_1$ but satisfying, for each n,

$$L_\theta(p_n) = p_{n-1}, \tag{3.4}$$

so that by continuity,

$$\lim_{n\to\infty} L(p_n) = L(p) \tag{3.5}$$

and at the same time

$$\lim_{n\to\infty} L(p_n) = \lim_{n\to\infty} p_{n-1} = p. \tag{3.6}$$

Thus $L(p) = p$, which would contradict the assumption that L_θ has exactly one fixed point. Therefore we can make the following definition.

Definition 3.2. Let θ be a cycle of order n such that L_θ has exactly one fixed point. The step number of θ, denoted $S(\theta)$, is the (smallest) value of i for which $E_i = \varnothing$.

Example 3.3. The cycle $\eta_1 = (1\ 2\ 3\ 4)$ has RL-pattern $RRRL$. The cycle $\eta_2 = (1\ 4\ 7\ 2\ 6\ 8\ 5)$ has RL-pattern $RRLRLRLL$; $R(\eta_1) = 3$ and $R(\eta_2) = 2$.

4. Results. For any cycle θ such that L_θ has exactly one fixed point, the following theorem describes a large class of cycles that cannot be forced by θ.

Theorem 4.1. *Let θ be a cycle of order $n \geq 2$ such that L_θ has exactly one fixed point. Let $S(\theta)$ denote the step number of θ. Let η be any cycle. If $R(\eta) > S(\theta)$, then θ does not force η.*

Proof. We have

$$1 < p_{S(\theta)} < p_{S(\theta)-1} < \cdots < p_2 < p_1 < n. \tag{4.1}$$

We write

$$[1,n] = \bigcup_{i=1}^{S(\theta)+1} I_i, \tag{4.2}$$

where

$$\begin{gathered} I_1 = [p_1, n], \\ I_i = [p_i, p_{i-1}] \quad \text{for } 2 \leq i \leq S(\theta), \\ I_{S(\theta)+1} = [1, p_{S(\theta)}]. \end{gathered} \tag{4.3}$$

For any $x \in \operatorname{int}(I_1)$, $L_\theta(x) < x$. So x cannot be the leftmost point in any periodic orbit. For $2 \leq i \leq S(\theta)+1$, we argue inductively. If $x \in \operatorname{int}(I_i)$, then $L_\theta(x) > x$ and $L_\theta(x) \in \bigcup_{j=1}^{i-1} I_j$, so if x is the leftmost point of a periodic orbit of type γ, the RL-pattern of γ consist of at most $i-1$ consecutive R's followed by an L. That is, $R(\gamma) \leq i-1$. This shows that any cycle η forced by θ must have $R(\eta) \leq S(\theta)$. □

Example 4.2. Let $\theta = (1\ 2\ 6\ 3\ 4\ 5)$. L_θ has exactly one fixed point and $S(\theta) = 3$. From Theorem 4.1, we know that for all $n \geq 5$, θ does not force $(1\ 2\ 3\ \cdots\ n)$. Using the technique developed in [8] it is seen that θ does force $(1\ 2\ 3\ 4)$ and that there are exactly two distinct orbits of type $(1\ 2\ 3\ 4)$. Also, θ forces $(1\ 2\ 3)$ and there are six distinct orbits of type $(1\ 2\ 3)$.

Example 4.3. Let $\theta = (1\ 3\ 5\ 2\ 8\ 4\ 7\ 6)$. L_θ has one fixed point and $S(\theta) = 2$. From Theorem 4.1, we see that for all $n \geq 4$, θ does not force $(1\ 2\ 3\ \cdots\ n)$. Using [8], one can find exactly two distinct orbits of type $(1\ 2\ 4\ 3)$, exactly fourteen distinct orbits of type $(1\ 3\ 2\ 4)$, exactly eleven distinct orbits of type $(1\ 4\ 2\ 3)$ and one can show that there are now orbits of type $(1\ 3\ 4\ 2)$ and no orbits of type $(1\ 4\ 3\ 2)$. These are the only orbit types of period 4 forced by θ.

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